UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

STABILITY OF SOLITARY WAVES FOR SOME SCHRÖDINGER–KDV SYSTEMS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

Degree of

DOCTOR OF PHILOSOPHY

By

SANTOSH BHATTARAI
Norman, Oklahoma
2013

STABILITY OF SOLITARY WAVES FOR SOME SCHRÖDINGER–KDV SYSTEMS

A DISSERTATION APPROVED FOR THE DEPARTMENT OF MATHEMATICS

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Acknowledgements

First, I would like to deeply thank my advisor, Professor John Albert, for his invaluable guidance and encouragement during my research and study at University of Oklahoma. This dissertation would have not been possible without his support and persistent help. He was always there to help me when I needed. I have been very privileged and proud to have him as my advisor.

I would like to thank sincerely my committee members : Dr. Boris Apanasov, Dr. Semion Gutman, and Dr. Rüdiger Landes for agreeing to serve in my committee, and for their advice and unconditional support during all these years of graduate school. I would also like to thank Dr. Musharraf Zaman, Associate Dean for Research and Graduate Programs, College of Engineering, who kindly accepted my request to be the outside member of my advisory committee. I would also like to thank Dr. Ralf Schmidt whose mentoring directly helped me to improve my knowledge. I would also like to thank OU Mathematics Department for providing me support during my graduate study.

I would like to thank my parents for their unconditional love and support since my childhood, and for encouraging me to pursue my dreams. I would also like to thank my brother and sisters for their support and encouraging me with their best wishes. I would like to give my special thanks to my wife, Puja Gautam, who was always there for me with her love, patient and continued support. Her tremendous support and encouragement enabled me to complete this work. Finally, I would like to express many thanks to my 5-month-old son, Nirvaan, for encouraging me with his own "oohs" and "aahs" language.

Contents

Abstract

This dissertation addresses existence and stability results for a two-parameter family of solitary-wave solutions to systems in which an equation of nonlinear Schrödinger type is coupled to an equation of Korteweg-de Vries type. Such systems govern interactions between long nonlinear waves and packets of short waves, and arises in fluid mechanics and plasma physics. Our proof involves the characterization of solitary-wave solutions as minimizers of an energy functional subject to two independent constraints. To establish the precompactness of minimizing sequence via concentrated compactness, we develop a new method of proving the sub-additivity of the problem with respect to both constraint variables jointly. The results extend the stability results previously obtained by Chen (1999), Albert and Angulo (2003), and Angulo (2006).

In addition, we also study the stability of solitary-wave solutions to an equation of short and long waves by using the techniques of convexity type. We shall apply the concentration compactness method to show the relative compactness of minimizing sequences for a different variational problem in which functional involved are not conserved quantities, and then, we use conserved quantities which arise from symmetries via Noether's theorem to obtain a relationship that makes it possible to utilize the variational properties of the solitary waves in the stability analysis. We prove that the stability of solitary waves is determined by the convexity of a function of the wave speed. The method is based on techniques appeared in Cazenave and Lions (1982), Levandosky (1998), and Angulo (2003), along with a convexity lemma of Shatah (1983).

Chapter 1

Introduction

1.1 Background

The central equations of study in this dissertation are model equations for waves which take account of both nonlinear and dispersive effects. In general, nonlinear effects become important when the waves being modelled have amplitudes large enough that the linear equations of motion are no longer good approximations on the time scales of interest. In particular, increasing the amplitude of a wave by multiplying it by a constant will affect the amplitude of the wave. Nonlinear effects tend to steepen the profile of a wave as it propagates. Dispersive effects become important when the medium through which the wave travels is such that the velocity of a wave is dependent on its frequency. They tend to cause the bulk of a wave to be dispersed as it propagates. The equations known as nonlinear dispersive wave equations are valid as models when these two types of effect are roughly of equal importance. Examples of such equations are the Korteweg-de Vries equation, Boussinesq equations, the Benjamin-Ono equation, and the nonlinear Schrödinger equation, along with many others. They appear as models for such varied phenomena as the propagation of pulses in long-distance communication devices such as transoceanic optical fibers ; atmospheric and oceanic internal and surface gravity waves ; elastic waves in the earth; and ion-acoustic waves in plasmas, to name but a few.

The equations mentioned above govern the time evolution of one-dimensional waves, and are written as equations for unknown functions $u(x,t)$ of a space variable x and a time variable t. Many of them have traveling-wave solutions, which are solutions of the form $u(x,t) = \phi(x - ct)$, representing a wave moving without change of shape, with constant velocity c. More generally, in the case when u is complex-valued, there also exist traveling wave solutions of the form $u(x,t) = e^{i\omega t}\phi(x - ct)$, where the phase velocity ω is a constant. In particular, when $\phi(z) \to 0$ as $z \to \infty$ and $z \to -\infty$, traveling waves are known as solitary waves. Intuitively, traveling waves and solitary waves occur when the competing effects of nonlinearity and dispersion are balanced. A typical feature of nonlinear dispersive wave equations is that such solutions exist for a range of values of the parameters ω and c, and play a significant role in the evolution of more general solutions of the underlying equations.

It is sometimes found that solitary waves retain their structure even after nonlinear interactions with other solitary waves. For example, two solitary waves with different velocities might effectively pass through each other without ultimately having an effect on each other, besides a change of phase. Solitary waves with such elastic scattering properties are generally known as solitons. The existence of solitons was first discovered in 1960's when they were brought to light as solutions of the Korteweg-de Vries equation. A detailed background on nonlinear waves and solitons may be found in [43], [21], [22], and [48].

In this dissertation our main interest will be in the stability of solitary-wave solutions of some nonlinear dispersive equations arising in mathematical physics. The equations we investigate do not appear, in general, to have soliton solutions which undergo completely elastic interactions. But we are able to show that they do have solitary-wave solutions which are stable in the sense that a slight perturbation of a solitary wave will continue to resemble a solitary wave for all time, rather than evolving into some other wave form. This stability property means that the solutions have the potential to model real, observable, physical phenomena.

We now introduce the equations that will be the focus of this dissertation. Both the nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^q u = 0 \tag{1.1}$$

for a complex-valued function u of $x \in \mathbb{R}$ and time t, and the generalized Korteweg-de Vries equation

$$v_t + v_{xxx} + v^p v_x = 0, \tag{1.2}$$

for a real-valued function v of x and t, are examples of universal models for nonlinear waves that describes many physical nonlinear systems. Equation (1.1) describes the evolution of small amplitude, slowly varying wave packets in nonlinear media. Equation (1.2) arises generically as a model for waves whose motion, to first order, is governed by the linear wave equation $v_t + v_x = 0$, but which on account of their long wavelength and small but finite amplitude are influenced by weak nonlinear and dispersive effects. Discussions of the canonical nature of these equations may be found, for example, in Chapters 13 and 17 of [48], Chapter 2 of [37], or Chapter 10 of [36].

In this dissertation we will consider a system describing the interaction of a nonlinear Schrödinger-type wave with a Korteweg-de Vries type wave:

$$\begin{aligned} iu_t + u_{xx} + \tau_1 |u|^q u &= -\alpha uv \\ v_t + v_{xxx} + \tau_2 v^p v_x &= -\frac{\alpha}{2}(|u|^2)_x, \end{aligned} \tag{1.3}$$

where τ_1, τ_2, and α are real constants. Systems of the form (1.3) govern the interactions between long waves and long-wavelength envelopes of short waves, and arise in fluid mechanics as well as in plasma physics. For example, it appears in [25] and [29] as a model for the interaction between long gravity waves and capillary waves on the surface of shallow water, in the case when the group velocity of the capillary wave coincides with the velocity of the long wave. In [7] and [38], a system of similar form appears as a model for the interaction of Langmuir waves and ion-acoustic waves in a plasma. A system of similar equations appears in [40] as the unidirectional reduction of a model for the resonant interaction of acoustic and optical modes in a diatomic lattice.

The system (1.3) possesses the following conserved quantities:

$$E(u,v) = \int_{-\infty}^{\infty} \left(|u_x|^2 + v_x^2 - \beta_1 |u|^{q+2} - \beta_2 v^{p+2} - \alpha |u|^2 v \right) \, dx, \tag{1.4}$$

where $\beta_1 = 2\tau_1/(q+2)$ and $\beta_2 = 2\tau_2/((p+1)(p+2))$,

$$G(u,v) = \int_{-\infty}^{\infty} v^2 \, dx + \mathrm{Im} \int_{-\infty}^{\infty} u \overline{u}_x \, dx, \tag{1.5}$$

where $\overline{u}_x$ is the complex conjugate of u_x and $\mathrm{Im}(z)$ denotes the imaginary part of z, and

$$H(u) = \int_{-\infty}^{\infty} |u|^2 \, dx. \tag{1.6}$$

In other words, for given initial functions $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$, the solution of (1.3) emanating from (u_0, v_0) has the property that

$$E(u(t), v(t)) = E(u_0, v_0), \;\; G(u(t), v(t)) = G(u_0, v_0) \text{ and } H(u(t)) = H(u_0)$$

for all t for which the solution exists.

This dissertation is concerned with the existence and stability results for (coupled) solitary traveling-wave solutions of (1.3). Such a solution is of the form

$$(u(x,t), v(x,t)) = \left(e^{i\omega t} e^{ic(x-ct)/2}\phi(x-ct), \psi(x-ct)\right), \tag{1.7}$$

where $c > 0$, $\omega \in \mathbb{R}$, and $\phi : \mathbb{R} \to \mathbb{C}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are functions that vanish at infinity in some sense (for example, $\phi \in H^1_{\mathbb{C}}$ and $\psi \in H^1$). (Here H^1 and $H^1_{\mathbb{C}}$ are L^2-based Sobolev spaces of real- and complex-valued functions on the line, respectively. For more details on our notation, see Section 1.3.) Inserting the ansatz (1.7) into (1.3), we see that (u, v) is a solution of (1.3) if and only if ϕ and ψ satisfy the system of ordinary differential equations

$$\begin{aligned} -\phi'' + \sigma\phi &= \tau_1 |\phi|^q \phi + \alpha\phi\psi \\ -\psi'' + c\psi &= \frac{\tau_2}{p+1}\psi^{p+1} + \frac{\alpha}{2}|\phi|^2, \end{aligned} \tag{1.8}$$

where $\sigma = \omega - c^2/4$, and primes denote derivatives of a function of a single variable.

We will use the following definition of stability throughout.

Definition 1.1. Let Y be a Banach space of ordered pairs of functions $(u(x), v(x))$ in which the initial value problem for equation (1.3) is well-posed. A subset $\mathcal{B}$ of Y is said to be stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(u_0, v_0) \in Y$ with

$$\inf_{(\phi,\psi)\in\mathcal{B}} \| (u_0, v_0) - (\phi, \psi) \| < \delta,$$

the solution $(u(x,t), v(x,t)$ of (1.3) with $(u(x,0), v(x,0)) = (u_0, v_0)$ exists for all t, and satisfies

$$\sup_{0\le t<\infty} \inf_{(\phi,\psi)\in\mathcal{B}} \|(u(.,t), v(.,t) - (\phi, \psi)\|_Y < \varepsilon.$$

By well-posedness in Y we generally mean that for given initial data in Y, there exists a time $T > 0$ and a unique solution (u, v) of (1.3), which stays in Y and depends continuously on the initial data as a map from Y to Y. For comprehensive discussion and bibliography of the topic of well-posedness of nonlinear dispersive wave equations, we refer the reader to the Dispersive PDE Wiki at http://wiki.math.toronto.edu/DispersiveWiki/.

1.2 Review of the Literature

The first rigorous treatment of the problem of stability of solitary-wave solutions to nonlinear dispersive equations was given by Benjamin [9] for the KdV solitary waves. Benjamin's arguments were improved and perfected by Bona [10]. Their theory uses the Hamiltonian structure of KdV equation and is based on the fact that solitary waves can be characterized as critical points of the Hamiltonian energy on level sets of a momentum functional.

Variational methods for proving orbital stability or instability of solitary-wave solutions to wave equations with Hamiltonian structure, based on the analysis of energy-momentum functionals, were subsequently greatly advanced by many authors. Notably, Grillakis, Shatah and Strauss [27] obtained sharp conditions for the orbital stability and instability of solitary waves for a class of abstract Hamiltonian systems. Bona, Souganidis and Strauss [11] obtained similar results for KdV type equations, a class not considered by Grillakis et al. [27]. For other important works in this direction, see Weinstein [46],[47], Shatah and Strauss [42], Maddocks and Sachs [35].

The stability theory of solitary-wave solutions developed in the works cited above rely on local analysis. This means that we must show that the solitary-wave solution is a local constrained minimizer of a Hamiltonian functional, and the procedure for this is carried out basically by studying specific spectral properties of a linear operator obtained by linearizing the solitary-wave equation. In practice this spectral analysis is difficult to carry out. An alternate method of proving stability of solitary waves, which avoids these difficulties, was developed by Cazenave and Lions [14] using the concentration-compactness principle of P. L. Lions [33]. In this approach, instead of starting with a given solitary wave and attempting to prove that it realizes a local minimum for a constrained variational problem, one starts with the constrained variational problem and looks for global minimizers. When the method works, it shows not only that global minimizers exist, but also that every minimizing sequence is relatively compact up to translations. This then is enough to conclude that the set of solitary waves which solve the minimization problem is a stable set. In [14], Cazenave and Lions proved the stability of solitary-wave solutions to nonlinear Schrödinger equations. In the last couple of decades, a similar method was applied by many authors to prove orbital stability of solitary waves for a great range of dispersive evolution equations: see for example, Albert [2], Albert et al. [4], Angulo [5], Chen et al. [17], Chen and Bona [16], Kichenassamy [30], and Ohta [39].

The work presented in Chapter 2 of this dissertation is in the same spirit as those above. We use the concentration compactness method to prove existence and stability results of solitary-wave solutions of (1.3). Our existence result is obtained by studying the variational problem of finding, for given positive values of s and t, minimizers of $E(u, v)$ subject to the constraints that $\int_{-\infty}^{\infty} |u|^2 \, dx = s$ and $\int_{-\infty}^{\infty} v^2 \, dx = t$. The connection to solitary waves is due to the fact that equations (1.8) are the Euler-Lagrange equations for this variational problem, with σ and c playing the role of Lagrange multipliers.

The standard technique of proving the stability of solitary-wave solutions using the concentration compactness method require proving the strict subadditivity of the variational problem with respect to the constraint parameters. More precisely, we require to prove strict subadditivity of the function $I(s,t)$ defined for $s > 0$ and $t > 0$ by

$$I(s,t) = \inf \left\{ E(f,g) : (f,g) \in Y, \int_{-\infty}^{\infty} |f|^2 \, dx = s, \text{ and } \int_{-\infty}^{\infty} g^2 \, dx = t \right\}. \quad (1.9)$$

For equations (1.1) or (1.2), the variational problems which characterize solitary waves depend on a single constraint parameter, and proofs of strict subadditivity are accomplished by simple arguments, dating back to Lions' original paper [33], which take advantage of homogeneities present in the equation.

To prove strict subadditivity for the two-parameter problem defined in (1.9), however, seems to be more difficult. In [3], which treats the case where $p = 1$ and $\tau_1 = 0$, it was noted that strict subadditivity, as defined below in Lemma 2.14, holds for $\alpha = 1/6$ (corresponding to setting the parameter q in [3] equal to 2), and it was shown that strict subadditivity continues to hold for α in some neighborhood of 1/6.

Here we are able to extend the existence result for solitary waves to all positive values of α, all non-negative values of τ_1, all positive valued of τ_2, all $p \in [1,4)$, and all $q \in [1,4)$. To do so, we prove subadditivity by relying on an argument due to Byeon [12] and Garrisi [26], which exploits the fact that the H^1 norms of certain functions are strictly decreased when the mass of the function is rearranged by symmetrization.

Previously, Dias et al. [20] had proved that for $p \in \{1,2,3\}$ (with $\alpha > 3$ if $p = 1$), (1.3) has an infinite family of positive bound states which decay exponentially at infinity. Compared to the result of [20], ours has the advantages that we do not require $\alpha > 3$ when $p = 1$, and also that the sets $\mathcal{S}_{s,t}$ of solitary waves obtained as minimizers of (1.9) form a true two-parameter family, in that $\mathcal{S}_{s_1,t_1}$ and $\mathcal{S}_{s_2,t_2}$ are disjoint if $(s_1,t_1) \neq (s_2,t_2)$. In [20], nonempty sets $\mathcal{T}_{\delta,\mu}$ of solitary waves are obtained by minimizing E subject to $\int |u|^2 + \delta v^2 = \mu$, but it is not clear whether $\mathcal{T}_{\delta_1,\mu_1}$ is necessarily disjoint from $\mathcal{T}_{\delta_2,\mu_2}$ if $(\delta_1,\mu_1) \neq (\delta_2,\mu_2)$.

Besides the question of existence of solitary-wave solutions of (1.8), a separate question we address in Chapter 2 is that of stability of these solitary-wave solutions as solutions of the initial-value problem for (1.3). The stability theory involves another variational characterization of solitary-wave solutions for (1.3). For $s > 0$ and $t \in \mathbb{R}$, define

$$W(s,t) = \inf\{E(h,g) : (h,g) \in Y,\ H(h) = s \text{ and } G(h,g) = t\}. \tag{1.10}$$

The variational problem associated to $W(s,t)$ is suitable for studying stability because not only the functional E being minimized, but also the constraint functionals G and H are conserved for (1.3). If minimizers (Φ,ψ) for $W(s,t)$ exist, they satisfy the Euler-Lagrange equations

$$\begin{aligned} -\Phi'' + \omega\Phi + ci\Phi' &= \tau_1|\Phi|^q\Phi + \alpha\Phi\psi \\ -\psi'' + c\psi &= \frac{\tau_2\psi^{p+1}}{p+1} + \frac{\alpha}{2}|\Phi|^2 \end{aligned} \tag{1.11}$$

where the real numbers c and ω are the Lagrange multipliers. These equations are satisfied by Φ and ψ if and only if the functions u and v defined by

$$(u(x,t), v(x,t)) = \left(e^{i\omega t}\Phi(x-ct), \psi(x-ct)\right) \tag{1.12}$$

are solutions of the NLS-KdV system (1.3). That is, solutions (Φ,ψ) of the variational problem for $W(s,t)$ are solitary-wave profiles, and (1.7) is recovered from (1.12) by setting $\Phi(x) = e^{icx/2}\phi(x)$. We use an argument given in Albert and Angulo [3] to prove the stability of solitary waves. Our stability theorem generalizes the stability results of [15], which treated the case when $\tau_1 = 0$, $p = 1$, and $\alpha = 1/6$; and of [3], which treated the case when $\tau_1 = 0$, $p = 1$, and α is in some neighborhood of 1/6. We also note the interesting paper of Angulo [5], which proves stability by a different method in the case when $\tau_1 = 0$, $p = 1$, $\alpha > 0$, and the wavespeed σ appearing in (1.8) is sufficiently small.

The approach presented in Chapter 2 for proving stability of solitary waves works whenever the functionals involved in the variational analysis are conserved quantities for the evolution equation in question. In Chapter 3, we show how the concentration compactness method can still be used to prove the stability of solitary waves if the functionals involved in the variational problem are not conserved quantities. This approach has been put forward by Levandosky, in [31], in which the stability of a fourth-order wave equation is studied. We apply this method to study the nonlinear stability of solitary-wave solutions of (1.3) with $p = 1$ and $q = 1$. We shall apply the concentration compactness method to show the relative compactness of minimizing sequences for a different variational problem that define solitary-wave solutions for (1.3) and then, we use functionals E, G, and H to obtain a relationship that makes it possible to utilize the variational properties of the traveling waves in the stability analysis. The proof of our stability result is based on the ideas of Cazenave and Lions [14], Levandosky [31], and Angulo [5], along with a convexity lemma of Shatah [41].

1.3 Dissertation outline

The dissertation is organized as follows. In Chapter 2, we prove existence and stability of a two-parameter family of solitary waves of (1.3). We begin by briefly discussing some well-posedness results which we will use in our stability analysis. For the benefit of the reader, we present an outline of the concentration compactness principle, which is the key tool in this dissertation. We refer the reader to the work of Albert [2] (see also [6]) to get a detailed illustration of the concentration compactness method, where the method is used to obtain stability results of solitary-wave solutions to nonlocal nonlinear wave equations. In Section 2.2, we prove a number of preparatory lemmas. We do not develop the elements of the theory of Sobolev spaces in this dissertation, but use a number of Sobolev type inequalities throughout the dissertation. A detailed account of Sobolev spaces can be found in Adams [1], Evans [23], Friedman [24], and Lieb and Loss [32]. Section 2.3 presents the concept of the symmetric decreasing rearrangement, which replaces a given nonnegative function f by a radial function f^*, and we prove Byeon and Garrisi's rearrangement lemma. In Section 2.4, we use rearrangement lemma of Section 2.3 to prove the strict subadditivity of $I(s,t)$. In Section 2.5, we use the concentration compactness method to prove the existence of solitary-wave solutions of (1.3). Finally, Section 2.6 provides the statement and proof of our stability theorem.

In Chapter 3, we study the stability of solitary-wave solutions of (1.3) with $p = 1$ and $q = 1$ by using the concentration compactness method and convexity techniques. In Section 3.2, by considering a different variational problem, we apply the concentration compactness method to prove the existence of solitary waves. In Section 3.3, after establishing some technical preliminaries, we use the conserved functionals of (1.3) to obtain a relationship that makes it possible to utilize the variational properties of the solitary waves in the stability analysis. We prove that the stability of solitary waves is determined by the convexity or concavity of a function of the wave speed.

Notation. The notation used in this dissertation is the standard notation used in the literature on partial differential equations. For $1 \le p \le \infty$, we denote by $L^p = L^p(\mathbb{R})$ the space of all measurable functions f on $\mathbb{R}$ for which the norm $|f|_p$ is finite, where

$$|f|_p = \left(\int_{-\infty}^{\infty} |f|^p \, dx \right)^{1/p} \quad \text{for } 1 \le p < \infty$$

and $|f|_\infty$ is the essential supremum of $|f|$ on $\mathbb{R}$. The Fourier transform $\widehat{f}$ of a tempered distribution $f(x) \in \mathcal{S}'(\mathbb{R})$ is defined as

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx.$$

For any tempered distribution f on $\mathbb{R}$ and any $s \in \mathbb{R}$, we define

$$\|f\|_s = \left(\int_{-\infty}^{\infty} \left(1 + |\xi|^2\right)^s \left| \widehat{f}(\xi) \right|^2 d\xi \right)^{1/2},$$

and we denote by $H^s_{\mathbb{C}}(\mathbb{R})$ the Sobolev space of all complex-valued functions f for which the norm $\|f\|_s$ is finite. We will always view $H^s_{\mathbb{C}}(\mathbb{R})$ as a vector space over the reals, with inner product given by

$$\langle f_1, f_2 \rangle = \mathrm{Re} \int_{-\infty}^{\infty} \left(1 + |\xi|^2\right)^s \widehat{f_1} \overline{\widehat{f_2}} \, d\xi.$$

The space of all real-valued functions f in $H^s_{\mathbb{C}}(\mathbb{R})$ will be denoted by $H^s(\mathbb{R})$. In particular, we use $\|f\|$ to denote the L^2 or $H^0(\mathbb{R})$ norm of a function f. We define the space Y to be $H^1_{\mathbb{C}}(\mathbb{R}) \times H^1(\mathbb{R})$, and the space X to be $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, each provided with the product norm.

We occasionally use below the operation of convolution of two functions, here denoted by the symbol $\star$ and defined by

$$f \star g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\, dy. \tag{1.13}$$

The letter C will frequently be used to denote various constants whose actual value is not important for our purposes.

Chapter 2

A Two-Parameter Family of Solitary-Wave Solutions to the Schrödinger-KdV Equations

In this chapter we prove existence and stability results of a two-parameter family of solitary waves of (1.3). We assume throughout the section, unless otherwise stated, that $\alpha > 0$, $\tau_1 \geq 0$, $\tau_2 > 0$, $1 \leq q < 4$, and $1 \leq p < 4$, where p is a rational number with odd denominator. Our proof involves the characterization of solitary-wave solutions as minimizers of an energy functional subject to two constraints. To establish the precompactness of minimizing sequences via concentrated compactness, we establish the sub-additivity of the problem with respect to both constraint variables jointly.

2.1 Introduction

The local and global well-posedness results of (1.3) have been studied by a large number of authors. For the non-periodic setting, the system of the form (1.3) with $p = 1$ and $q = 2$ was first studied in Tsutsumi [45] for a global well-posedness theory in $H_{\mathbb{C}}^{m+1/2}(\mathbb{R}) \times H_{\mathbb{R}}^{m}(\mathbb{R})$ with $m \in \mathbb{N}$. Later, by using the Fourier restriction method, Bekiranov, Ogawa and Ponce [8] proved a local theory in $H_{\mathbb{C}}^{s}(\mathbb{R}) \times H_{\mathbb{R}}^{s-1/2}(\mathbb{R})$ for $s \geq 0$. Corcho and Linares [18] proved that the system (1.3) with $p = 1$ and $q = 2$ is locally well-posed for initial data

$(u_0, v_0) \in H^k(\mathbb{R}) \times H^s(\mathbb{R})$ with $k \geq 0,\ s > -3/4$ and

$$
\begin{aligned}
k-1 &\leq s \leq 2k-1/2 \ \text{ if } k \leq 1/2, \\
k-1 &\leq s \leq k+1/2 \quad \text{if } k > 1/2.
\end{aligned}
$$

Furthermore, they proved the global well-posedness of the Cauchy problem associated to (1.3) in the energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ by using three conserved quantities discovered by Tsutsumi [45]. (See also Guo and Miao [28] for a well-posedness result for $q = 2$). Dias, Figueira, and Oliveira [20] recently proved the existence of an infinite family of smooth positive bound states for (1.3) which decay exponentially at infinity.

In our study of existence and stability of solitary-wave solutions of (1.3), we use the method of concentration compactness to prove the relative compactness of minimizing sequences for the variational problem, and hence the existence of minimizers. The method is based on the following lemma:

Theorem 2.1 (Lions [33]). *Let $\{\rho_n\}_{n\geq 1}$ be a sequence of nonnegative functions in $L^1(\mathbb{R})$ satisfying $\int_{-\infty}^{\infty} \rho_n(x)\ dx = \lambda$ for all n and some $\lambda > 0$. Then there exists a subsequence $\{\rho_{n_k}\}_{k\geq 1}$ satisfying one of the following three conditions:*

(1) (Compactness) There are $y_k \in \mathbb{R}$ for $k = 1, 2, \ldots$, such that $\rho_{n_k}(. + y_k)$ is tight, i.e., for any $\varepsilon > 0$, there is $R > 0$ large enough such that

$$\int_{|x-y_k|\leq R} \rho_{n_k}(x)\ dx \geq \lambda - \varepsilon.$$

(2) (Vanishing) For any $R > 0$,

$$\lim_{k\longrightarrow\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\leq R} \rho_{n_k}(x)\ dx = 0.$$

(3) (Dichotomy) There exists $\alpha \in (0, \lambda)$ such that for any $\varepsilon > 0$, there exists $k_0 \geq 1$ and $\rho_k^1, \rho_k^2 \in L^1(\mathbb{R})$, with $\rho_k^1, \rho_k^2 \geq 0$, such that for $k \geq k_0$,

$$\begin{cases} |\rho_{n_k} - (\rho_k^1 + \rho_k^2)|_1 \leq \varepsilon, \quad \left|\int_{-\infty}^{\infty} \rho_k^1 \, dx - \alpha\right| \leq \varepsilon, \quad \left|\int_{-\infty}^{\infty} \rho_k^2 \, dx - (\lambda - \alpha)\right| \leq \varepsilon, \\ \quad supp(\rho_k^1) \cap supp(\rho_k^2) = \emptyset, \quad dist(supp(\rho_k^1), supp(\rho_k^2)) \to \infty \quad as \ k \to \infty. \end{cases}$$

Remark 2.2. In Theorem 2.1 above, the condition $\int_{-\infty}^{\infty} \rho_n(x) \, dx = \lambda$ can be replaced by $\int_{-\infty}^{\infty} \rho_n(x) \, dx = \lambda_n$ where $\lambda_n \to \lambda > 0$ as $n \to \infty$. Indeed, it is enough to replace ρ_n by ρ_n/λ_n and apply the theorem.

Typically, one proves compactness by ruling out the last two possibilities. This requires proving the strict subadditivity of the function $I(s,t)$. In the next few sections we will focus on proving the strict subadditivity of $I(s,t)$.

2.2 The two-parameter variational problem

We consider the problem of finding, for any $s, t > 0$,

$$I(s,t) = \inf \left\{ E(f,g) : (f,g) \in Y, \ \|f\|^2 = s \text{ and } \|g\|^2 = t \right\}, \tag{2.1}$$

where $E(f,g)$ is defined by (1.4). We define a minimizing sequence for $I(s,t)$ to be any sequence $\{(f_n, g_n)\}$ of functions in Y satisfying

$$\lim_{n\to\infty} \|f_n\|^2 = s, \ \lim_{n\to\infty} \|g_n\|^2 = t, \text{ and } \lim_{n\to\infty} E(f_n, g_n) = I(s,t). \tag{2.2}$$

Lemma 2.3. *Every minimizing sequence for $I(s,t)$ is bounded in Y. Furthermore, one has $-\infty < I(s,t) < 0$.*

Proof. First, observe that if $\{(f_n, g_n)\}$ is a minimizing sequence for $I(s,t)$, then $\|f_n\|$ and $\|g_n\|$ are bounded. From the Gagliardo-Nirenberg inequality (see, for example, Theorem 9.3 of [24]), we have that

$$|f_n|_{q+2}^{q+2} \le C\|f_{nx}\|^{q/2}\|f_n\|^{(q+4)/2}, \tag{2.3}$$

and since $\|f_n\|$ is constant, it follows that

$$|f_n|_{q+2}^{q+2} \le C\|(f_n, g_n)\|_Y^{q/2}. \tag{2.4}$$

Similarly,

$$|g_n|_{p+2}^{p+2} \le C\|g_{nx}\|^{p/2} \le C\|(f_n, g_n)\|_Y^{p/2}. \tag{2.5}$$

(Here, as throughout the paper, C denotes various constants which may depend on s and t but are independent of f_n and g_n.) Moreover, the same estimate (2.4) with q replaced by 2 shows that

$$|f_n|_4^4 \le C\|f_{nx}\| \cdot \|f_n\|^3 \le C\|f_{nx}\|,$$

so by Hölder's inequality,

$$\int_{-\infty}^{\infty} |f_n|^2|g_n|\ dx \le |f_n|_4^2 \cdot \|g_n\| \le C\|f_{nx}\|^{1/2} \le C\|(f_n, g_n)\|_Y^{1/2}. \tag{2.6}$$

Now

$$\begin{aligned}
&\|(f_n, g_n)\|_Y^2 = \|f_n\|_1^2 + \|g_n\|_1^2 \\
&\quad = E(f_n, g_n) + \int_{-\infty}^{\infty} \left(\beta_1|f_n|^{q+2} + \beta_2 g_n^{p+2} + \alpha|f_n|^2 g_n\right)\ dx + \|f_n\|^2 + \|g_n\|^2,
\end{aligned}$$

and $E(f_n, g_n)$ is bounded since $\{(f_n, g_n)\}$ is a minimizing sequence. Therefore from (2.4), (2.5), and (2.6) it follows that

$$\|(f_n, g_n)\|_Y^2 \leq C\left(1 + \|(f_n, g_n)\|_Y^{1/2} + \|(f_n, g_n)\|_Y^{q/2} + \|(f_n, g_n)\|_Y^{p/2}\right).$$

Since $q/2 < 2$ and $p/2 < 2$, we deduce that $\|(f_n, g_n)\|_Y$ is bounded.

Once we have shown that $\{(f_n, g_n)\}$ is bounded in Y, a finite lower bound on $E(f_n, g_n)$ also follows immediately from (2.4), (2.5), and (2.6). So $I(s,t) > -\infty$.

Finally, to see that $I(s,t) < 0$, choose $(f,g) \in Y$ such that $\|f\|^2 = s$, $\|g\|^2 = t$, and $f(x) > 0$ and $g(x) > 0$ for all $x \in \mathbb{R}$. For each $\theta > 0$, the functions $f_\theta(x) = \theta^{1/2} f(\theta x)$ and $g_\theta(x) = \theta^{1/2} g(\theta x)$ satisfy $\|f_\theta\|^2 = s$, $\|g_\theta\|^2 = t$, and

$$\begin{aligned} E(f_\theta, g_\theta) &= \int_{-\infty}^{\infty} \left(|f_{\theta x}|^2 + g_{\theta x}^2 - \beta_1 |f_\theta|^{q+2} - \beta_2 g_\theta^{p+2} - \alpha |f_\theta|^2 g_\theta\right)\, dx \\ &\leq \theta^2 \int_{-\infty}^{\infty} \left(|f_x|^2 + g_x^2\right)\, dx - \theta^{1/2} \int_{-\infty}^{\infty} \alpha |f|^2 g\, dx. \end{aligned}$$

Hence, by taking θ sufficiently small, we get $E(f_\theta, g_\theta) < 0$, proving that $I(s,t) < 0$. □

Lemma 2.4. *Suppose (f_n, g_n) is a minimizing sequence for $I(s,t)$, where $t > 0$ and $s \geq 0$. (Note that we do not require $s > 0$ here.) Then there exists $\delta > 0$ such that $\|g_{nx}\| \geq \delta$ for all sufficiently large n.*

Proof. If the conclusion is not true, then by passing to a subsequence we may assume there exists a minimizing sequence for which $\lim_{n\to\infty} \|g_{nx}\| = 0$. From (2.5) it then follows that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n^{p+2}\, dx = 0.$$

Moreover, because of the elementary estimate

$$|g_n|_\infty \leq C \|g_n\|^{1/2} \|g_{nx}\|^{1/2},$$

we can write, in place of (2.6),

$$\int_{-\infty}^{\infty} |f_n|^2 |g_n| \ dx \le C\|f_n\|^2 \|g_n\|^{1/2} \|g_{nx}\|^{1/2} \le C\|g_{nx}\|^{1/2}, \tag{2.7}$$

from which it follows that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n|^2 g_n \ dx = 0.$$

Hence

$$\begin{aligned} I(s,t) &= \lim_{n\to\infty} E(f_n, g_n) \\ &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1 |f_n|^{q+2}\right) \ dx. \end{aligned} \tag{2.8}$$

Now let ψ be any non-negative function such that $\|\psi\|^2 = t$. For every $\theta > 0$, the function $\psi_\theta(x) = \theta^{1/2}\psi(\theta x)$ satisfies $\|\psi_\theta\|^2 = t$, so that $I(s,t) \le E(f_n, \psi_\theta)$ for all n. On the other hand, if we define

$$\eta = \theta^2 \int_{-\infty}^{\infty} \psi_x^2 \ dx - \beta_2 \theta^{p/2} \int_{-\infty}^{\infty} \psi^{p+2} \ dx, \tag{2.9}$$

then since $p/2 < 1$, by fixing $\theta > 0$ sufficiently small we can arrange that

$$\eta < 0. \tag{2.10}$$

Then for all $n \in \mathbb{N}$,

$$\begin{aligned} I(s,t) &\le E(f_n, \psi_\theta) \\ &= \int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1 |f_n|^{q+2} - \theta^{1/2}\alpha |f_n|^2 \psi\right) \ dx + \eta \\ &\le \int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1 |f_n|^{q+2}\right) \ dx + \eta. \end{aligned}$$

Therefore

$$I(s,t) \leq \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1 |f_n|^{q+2}\right)\ dx + \eta,$$

which contradicts (2.8) and (2.10). $\square$

Lemma 2.5. *Suppose $g(x)$ is an integrable function on $\mathbb{R}$ such that*

$$\int_{-\infty}^{\infty} g(x)\ dx > 0. \tag{2.11}$$

Then for every $s > 0$ there exists $f \in H^1$ such that $\|f\|^2 = s$ and

$$\int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g\right)\ dx < 0.$$

Proof. Let ψ be an arbitrary smooth, non-negative function with compact support such that $\psi(0) = 1$ and $\|\psi\|^2 = s$, and for $\theta > 0$ define $\psi_\theta(x) = \theta^{1/2}\psi(\theta x)$. Then $\|\psi_\theta\|^2 = s$, and

$$\int_{-\infty}^{\infty} \left(\psi_{\theta x}^2 - \psi_\theta^2 g\right)\ dx = \theta^2 \int_{-\infty}^{\infty} \psi_x^2\ dx - \theta \int_{-\infty}^{\infty} \psi(\theta x)^2 g(x)\ dx. \tag{2.12}$$

But, by the Dominated Convergence Theorem,

$$\lim_{\theta\to 0} \int_{-\infty}^{\infty} \psi(\theta x)^2 g(x)\ dx = B,$$

where $B = \displaystyle\int_{-\infty}^{\infty} g(x)\ dx > 0$. Therefore from (2.12) it follows that

$$\int_{-\infty}^{\infty} \left(\psi_{\theta x}^2 - \psi_\theta^2 g\right)\ dx \leq \theta^2 \int_{-\infty}^{\infty} \psi_x^2\ dx - \theta B/2 \tag{2.13}$$

for all θ in some neighborhood of 0. Since the quantity on the right-hand side can be made negative by taking θ sufficiently small, the desired f can be found by taking $f = \psi_\theta$ for a sufficiently small value of θ. $\square$

Lemma 2.6. *Define $J : H^1 \to \mathbb{R}$ by*

$$J(g) = \int_{-\infty}^{\infty} \left(g_x^2 - \beta_2 g^{p+2} \right) \, dx. \tag{2.14}$$

Let $t > 0$, and let $\{g_n\}$ be any sequence of functions in H^1 such that

$$\lim_{n \to \infty} \|g_n\|^2 = t,$$

and

$$\lim_{n \to \infty} J(g_n) = \inf \, \left\{ J(g) : g \in H^1 \text{ and } \|g\|^2 = t \right\}.$$

Then there exists a subsequence $\{g_{n_k}\}$ and a sequence of real numbers y_k such that $g_{n_k}(x + y_k)$ converges strongly in H^1 norm to $g_0(x)$, where

$$g_0(x) = \left(\frac{\lambda}{\beta_2} \right)^{1/p} \operatorname{sech}^{2/p} \left(\frac{\sqrt{\lambda} p x}{2} \right), \tag{2.15}$$

and $\lambda > 0$ is chosen so that $\|g_0\|^2 = t$. In particular,

$$J(g_0) = \inf \, \left\{ J(g) : g \in H^1 \text{ and } \|g\|^2 = t \right\}. \tag{2.16}$$

Proof. The proof that some subsequence of g_n must converge, after suitable translations, strongly in H^1 norm is by now a standard exercise in the use of the method of concentration compactness. A proof in the case $p = 1$ appears, for example, in Theorem 2.9 of [2], or Theorem 3.13 of [3]. A similar proof, with obvious alterations, works for all $p \in [1, 4)$ because for such p the Gagliardo-Nirenberg inequality (2.5) permits one to obtain a uniform bound on $\|g_n\|_1$.

Denote the translated subsequence of $\{g_n\}$ which converges strongly by $\{g_{n_k}(x+\tilde{y}_k)\}$, and let $\psi \in H^1$ be its limit. Then ψ must satisfy

$$J(\psi) = \inf \left\{ J(g) : g \in H^1 \text{ and } \|g\|^2 = t \right\}, \tag{2.17}$$

and must also be a solution of the Euler-Lagrange equation

$$-2\psi'' - (p+2)\beta_2 \psi^{p+1} = -2\lambda\psi \tag{2.18}$$

for some real number λ. Equation (2.18) can be explicitly integrated to show that, in order for ψ to be in H^1, λ must be positive and ψ must be a translate of the function g_0 defined in (2.15), say $\psi(x) = g_0(x+y_0)$ for some $y_0 \in \mathbb{R}$. Then (2.16) follows from (2.17). Also, defining $y_k = \tilde{y}_k - y_0$, we have that $g_{n_k}(x+y_k)$ converges to g_0 in H^1. $\square$

Lemma 2.7. *Suppose $\beta_1 > 0$, and define $\tilde{J} : H^1_{\mathbb{C}} \to \mathbb{R}$ by*

$$\tilde{J}(f) = \int_{-\infty}^{\infty} \left(|f_x|^2 - \beta_1 |f|^{q+2} \right) \, dx. \tag{2.19}$$

Let $s > 0$, and let $\{f_n\}$ be any sequence of functions in $H^1_{\mathbb{C}}$ such that

$$\lim_{n\to\infty} \|f_n\|^2 = s,$$

and

$$\lim_{n\to\infty} \tilde{J}(f_n) = \inf \left\{ \tilde{J}(f) : f \in H^1_{\mathbb{C}} \text{ and } \|f\|^2 = s \right\}.$$

Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$, a sequence of real numbers y_k, and a real number θ such that $e^{-i\theta} f_{n_k}(x+y_k)$ converges strongly in $H^1_{\mathbb{C}}$ norm to

$f_0(x)$, where

$$f_0(x) = \left(\frac{\lambda}{\beta_1}\right)^{1/q} \operatorname{sech}^{2/q}\left(\frac{\sqrt{\lambda} p x}{2}\right), \tag{2.20}$$

and $\lambda > 0$ is chosen so that $\|f_0\|^2 = s$. In particular,

$$\tilde{J}(f_0) = \inf\ \left\{\tilde{J}(f) : f \in H^1_{\mathbb{C}} \text{ and } \|f\|^2 = s\right\}. \tag{2.21}$$

Proof. The comments in the first paragraph of the proof of Lemma 2.6 apply as well to $\tilde{J}$ as to J, since the proof alluded to there works here with no formal changes: the only difference is that now $\|f_n\|$ represents the modulus of a complex-valued function. Therefore we can conclude that there exists a subsequence $\{f_{n_k}\}$ and a sequence of real numbers $\tilde{y}_k$ such that $\{f_{n_k}(x + \tilde{y}_k)\}$ converges strongly in $H^1_{\mathbb{C}}$ to a (now complex-valued) function ϕ for which

$$\tilde{J}(\phi) = \inf\ \left\{J(f) : f \in H^1_{\mathbb{C}} \text{ and } \|f\|^2 = t\right\}, \tag{2.22}$$

and for which the Euler-Lagrange equation

$$-2\phi'' - (q+2)\beta_1 \phi^{q+1} = -2\lambda\phi \tag{2.23}$$

holds, where here λ is again a real number.

It is proved in Theorem 8.1.6 of [13] that for every solution ϕ of (2.23), there exists a real number θ such that $\phi(x) = e^{i\theta}\tilde{\phi}(x)$ on $\mathbb{R}$, where $\tilde{\phi}(x)$ is real-valued and positive (the same argument used there is also given below in the proof of part (iv) of Theorem 2.15). The H^1 function $\tilde{\phi}$ also satisfies (2.23), and so, as in the proof of Lemma 2.6, it follows that there exists $y_0 \in \mathbb{R}$ such that $\tilde{\phi}(x) = f_0(x + y_0)$ on $\mathbb{R}$, where f_0 is as defined in (2.20). Since $\tilde{J}(\phi) = \tilde{J}(\tilde{\phi})$, then (2.21) follows from (2.22). Also, if we define $y_k = \tilde{y}_k - y_0$, then we have that $e^{-i\theta} f_{n_k}(x + y_k)$ converges in $H^1_{\mathbb{C}}$ to f_0. $\square$

Lemma 2.8. *Suppose (f_n, g_n) is a minimizing sequence for $I(s,t)$, where $s > 0$ and $t \geq 0$. If $t > 0$, or $t = 0$ and $\beta_1 > 0$, then there exists $\delta > 0$ such that $\|f_{nx}\| \geq \delta$ for all sufficiently large n. If $t = 0$ and $\beta_1 = 0$, then $I(s,t) = 0$.*

Proof. As in the proof of Lemma (2.4), we argue by contradiction. If the conclusion is not true, then by passing to a subsequence we may assume there exists a minimizing sequence for which $\lim_{n\to\infty} \|f_{nx}\| = 0$. From (2.3) and (2.6) we have that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n|^2 g_n = \lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n|^{q+2} = 0, \tag{2.24}$$

so

$$I(s,t) = \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(g_{nx}^2 - \beta_2 g_n^{p+2}\right)\, dx. \tag{2.25}$$

In case $t > 0$, we have from (2.16) that

$$I(s,t) \geq J(g_0), \tag{2.26}$$

where g_0 is as (2.15), and therefore g_0 is integrable with positive integral. Therefore, by Lemma 2.5 there exists $f \in H^1$ such that $\|f\|^2 = s$ and

$$\int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g_0\right)\, dx < 0. \tag{2.27}$$

It follows that

$$I(s,t) \leq E(f, g_0) = \int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g_0 - \beta_1 |f|^{q+2}\right)\, dx + J(g_0) < J(g_0), \tag{2.28}$$

which contradicts (2.26).

In case $t = 0$ and $\beta_1 > 0$, then by (2.25), $I(s,t) = 0$. On the other hand

$I(s,t) = I(s,0)$ is the infimum of

$$E(f,0) = \int_{-\infty}^{\infty} \left(|f_x|^2 - \beta_1 |f|^{q+2}\right)\ dx \tag{2.29}$$

over all $f \in H^1_{\mathbb{C}}$ satisfying $\|f\|^2 = s$. Let f be any non-negative function in H^1 such that $\|f\|^2 = s$, and define $f_\theta(x) = \theta^{1/2} f(\theta x)$. Then

$$E(f_\theta, 0) = \theta^2 \int_{-\infty}^{\infty} f_x^2\ dx - \beta_1 \theta^{q/2} \int_{-\infty}^{\infty} f^{q+2}\ dx, \tag{2.30}$$

and since $q < 4$, we can make the right-hand side negative by choosing a sufficiently small value of θ. Therefore $I(s,t) < 0$, giving a contradiction.

Finally, if $t = 0$ and $\beta_1 = 0$, then $I(s,t) = I(s,0)$ is the infimum of

$$E(f,0) = \int_{-\infty}^{\infty} |f_x|^2\ dx \tag{2.31}$$

over all f in $H^1_{\mathbb{C}}$ such that $\|f\|^2 = s$. This infimum is clearly non-negative, but on the other hand if we replace f by f_θ, as defined in the preceding paragraph, then we can make $E(f_\theta, 0)$ arbitrarily small by taking θ sufficiently small. Hence $I(s,t) = 0$. $\square$

Lemma 2.9. *Suppose (f_n, g_n) is a minimizing sequence for $I(s,t)$, where $s > 0$ and $t > 0$. Then there exists $\delta > 0$ such that for all sufficiently large n,*

$$\int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1 |f_n|^{q+2} - \alpha |f_n|^2 g_n\right)\ dx \le -\delta.$$

Proof. If the conclusion is false, then by passing to a subsequence we may assume that there exists a minimizing sequence (f_n, g_n) for which

$$\liminf_{n\to\infty} \int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1 |f_n|^{q+2} - \alpha |f_n|^2 g_n\right)\ dx \ge 0, \tag{2.32}$$

and so

$$I(s,t) = \lim_{n\to\infty} E(f_n, g_n) \geq \liminf_{n\to\infty} \int_{-\infty}^{\infty} \left(g_{nx}^2 - \beta_2 g_n^{p+2}\right)\ dx. \tag{2.33}$$

Define J and g_0 as in Lemma 2.6. Then (2.33) implies that

$$I(s,t) \geq J(g_0). \tag{2.34}$$

On the other hand, by Lemma 2.5, there exists $f \in H^1$ such that $\|f\|^2 = s$ and

$$\int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g_0\right)\ dx < 0.$$

Therefore

$$I(s,t) \leq E(f, g_0) \leq \int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g_0\right)\ dx + J(g_0) < J(g_0), \tag{2.35}$$

which contradicts (2.34). $\square$

Lemma 2.10. *For all $(f,g) \in Y$, one has $E(|f|,|g|) \leq E(f,g)$.*

Proof. What has to be proved is that if $f \in H^1_{\mathbb{C}}$, then $|f(x)|$ is in H^1 and

$$\int_{-\infty}^{\infty} \left||f|_x\right|^2\ dx \leq \int_{-\infty}^{\infty} |f_x|^2\ dx. \tag{2.36}$$

For the reader's convenience, we repeat the proof from Albert et al. [4]. Let $\mu > 0$, and define the function $N_\mu(x)$ by $\widehat{N_\mu}(\xi) = 1/(\mu + \xi^2)$. Then $N_\mu(x) > 0$ for all $x \in \mathbb{R}$. Moreover $N_\mu \in L^p$ for every $p \in [1,\infty]$. Now, if $g = |f|$, then $N_\mu * g(x) \geq N_\mu * f(x)$ for all $x \in \mathbb{R}$ and every $\mu > 0$. In consequence, one has that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{\mu+\xi^2} |\widehat{g}(\xi)|^2 \, d\xi &= \int_{-\infty}^{\infty} g(x)(N_\mu * g)(x) \, dx \\ &\geq \int_{-\infty}^{\infty} f(x)(N_\mu * f)(x) \, dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\mu+\xi^2} \left| \widehat{f}(\xi) \right|^2 \, d\xi.\end{aligned}$$

By Parseval's identity, $\int_{-\infty}^{\infty} |\widehat{g}(\xi)|^2 \, d\xi = \int_{-\infty}^{\infty} | \widehat{f}(\xi) |^2 \, d\xi$, so it follows that

$$\int_{-\infty}^{\infty} \mu \left[1 - \frac{\mu}{\mu+\xi^2}\right] \left| \widehat{f}(\xi) \right|^2 \, d\xi \geq \int_{-\infty}^{\infty} \mu \left[1 - \frac{\mu}{\mu+\xi^2}\right] |\widehat{g}(\xi)|^2 \, d\xi.$$

Taking the limit $\mu \to \infty$ on both sides of the preceding inequality, and using the monotone convergence theorem gives

$$\int_{-\infty}^{\infty} |\xi|^2 \left| \widehat{f}(\xi) \right|^2 \, d\xi \geq \int_{-\infty}^{\infty} |\xi|^2 |\widehat{g}(\xi)|^2 \, d\xi,$$

which yields the desired result. □

We end this section with the following lemma.

Lemma 2.11. *The functionals E, G, and H are continuous from Y to $\mathbb{R}$.*

Proof. This follows easily (for all $p \geq 0$ and $q \geq 0$) from the Sobolev embedding theorem, in particular using the fact that the inclusion of H^1 in L^∞ is continuous.

□

2.3 Symmetrization and a technical lemma

The concept of the *symmetric rearrangement* of a function will play an important role in our proof of strict subadditivity of the function $I(s,t)$. For a non-negative function $w : \mathbb{R} \to [0, \infty)$, if $\{x : w(x) > y\}$ has finite measure $m(w, y)$ for all $y > 0$, then the symmetric decreasing rearrangement w^* of w is defined by

$$w^*(x) = \inf \{y \in (0, \infty) : \frac{1}{2} m(w, y) \leq x\} \tag{2.37}$$

(or see page 80 of [32] for a different but equivalent definition). For (f, g) in Y, both $|f|$ and $|g|$ are in H^1, and hence $|f|^*$ and $|g|^*$ are well-defined.

The next lemma state that $E(f, g)$ decreases when f and g are replaced by $|f|$ and $|g|$, and when $|f|$ and $|g|$ are symmetrically rearranged.

Lemma 2.12. *For all $(f, g) \in Y$, one has $E(|f|^*, |g|^*) \leq E(f, g)$.*

Proof. This follows from classic estimates on the symmetric rearrangements of functions. A basic fact about rearrangements is that they preserve L^p norms (cf. page 81 of [32]), so that

$$\int_{-\infty}^{\infty} (|f|^*)^{q+2} \, dx = \int_{-\infty}^{\infty} |f|^{q+2} \, dx \tag{2.38}$$

and

$$\int_{-\infty}^{\infty} (|g|^*)^{p+2} \, dx = \int_{-\infty}^{\infty} |g|^{p+2} \, dx. \tag{2.39}$$

Another basic inequality about rearrangements, Theorem 3.4 of [32], implies that

$$\int_{-\infty}^{\infty} (|f|^*)^2 |g|^* \, dx \geq \int_{-\infty}^{\infty} |f|^2 |g| \, dx. \tag{2.40}$$

Finally, from Lemma 7.17 of [32] we have that

$$\int_{-\infty}^{\infty} |(|f|^*)_x|^2 \, dx \leq \int_{-\infty}^{\infty} ||f|_x|^2 \, dx,$$

and similarly for $g(x)$. In light of these facts, and because α, β_1, and β_2 are all non-negative, it follows from Lemma 2.10 that $E(|f|^*, |g|^*) \leq E(f, g)$.

$\square$

We will also make crucial use of the following Lemma, due to Garrisi [26] (see also the N-dimensional version given in Byeon [12]). We include a proof here since our version of the lemma differs slightly from that stated by Garrisi.

Lemma 2.13. *Suppose u and v are non-negative, even, C^∞ functions with compact support in $\mathbb{R}$, which are non-increasing on $\{x : x \geq 0\}$. Let x_1 and x_2 be numbers such that $u(x + x_1)$ and $v(x + x_2)$ have disjoint supports, and define*

$$w(x) = u(x + x_1) + v(x + x_2).$$

Let $w^* : \mathbb{R} \to \mathbb{R}$ be the symmetric decreasing rearrangement of w. Then the distributional derivative $(w^*)'$ of w^* is in L^2, and satisfies

$$\|(w^*)'\|^2 \leq \|w'\|^2 - \frac{3}{4} \min\{\|u'\|^2, \|v'\|^2\}. \tag{2.41}$$

Proof. First consider the case when $u'(x) < 0$ for all $x \in (0, c)$ and $v'(x) < 0$ for all $x \in (0, d)$, where $[-c, c]$ is the support of u and $[-d, d]$ is the support of v. Let $a = \sup\{u(x) : x \in \mathbb{R}\}$ and $b = \sup\{v(x) : x \in \mathbb{R}\}$. By interchanging u and v if necessary, we may assume that $a \leq b$.

Define $z_u : [0, \infty) \to [0, c]$ by

$$z_u(y) = \inf\{x \in [0, \infty) : u(x) \leq y\}. \tag{2.42}$$

For $y \in (0, a)$, $z_u(y)$ is equal to the unique number $x(y) \in (0, c)$ such that

$u(x(y)) = y$. The function z_u is differentiable on $(0, a)$, with derivative

$$z_u'(y) = \frac{1}{u'(x(y))} < 0,$$

and we have

$$\begin{aligned} \|u'\|^2 &= 2\int_0^c (u'(x))^2 \, dx \\ &= 2\int_0^a \frac{-1}{z_u'(y)} \, dy \\ &= 2\int_0^a \frac{1}{|z_u'(y)|} \, dy. \end{aligned}$$

For $y \geq a$ we have $z_u(y) = 0$.

Similarly, we define $z_v : [0, \infty) \to [0, d]$ by

$$z_v(y) = \inf\{x \in [0, \infty) : v(x) \leq y\}. \tag{2.43}$$

Then

$$y_v'(v(x)) = \frac{1}{v'(x)} < 0$$

on $(0, d)$, and

$$\|v'\|^2 = 2\int_0^b \frac{1}{|z_v'(y)|} \, dy.$$

Now, for each $y \in [0, \infty)$, define

$$z(y) = z_u(y) + z_v(y). \tag{2.44}$$

Then z is continuous on $[0, \infty)$ and differentiable, with strictly negative derivative, on $(0, a)$ and on (a, b). Therefore z is strictly decreasing on $[0, b]$, and so its restriction to $[0, b]$ has an inverse function $z^{-1} : [0, c + d] \to [0, b]$, with $z^{-1}([0, c]) = [a, b]$ and $z^{-1}([c, c + d]) = ([0, a])$. From (2.37) and the definition of w, using the fact that $u(x + x_1)$ and $v(x + x + 2)$ have disjoint supports, we see that w^* is supported on $[0, c+d]$ and coincides with z^{-1} there. In particular, for all $y \in (0, a) \cup (a, b)$, we have

$$(w^*)'(z(y)) = \frac{1}{z_u'(y) + z_v'(y)}.$$

Now making use of the fact that for all positive numbers μ and ν, there holds the elementary inequality

$$\frac{2}{\mu+\nu} \leq \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{\nu}\right),$$

we have the following computation:

$$\begin{aligned}
\|(w^*)'\|^2 &= 2\int_0^{c+d} ((w^*)'(x))^2 \, dx \\
&= 2\int_0^{c} ((w^*)'(x))^2 \, dx + 2\int_c^{c+d} ((w^*)'(x))^2 \, dx \\
&= 2\int_0^{a} \frac{1}{|z_u'(y)| + |z_v'(y)|} \, dy + 2\int_a^{b} \frac{1}{|z_v'(y)|} \, dy \\
&\leq \frac{1}{2}\int_0^{a} \left(\frac{1}{|z_u'(y)|} + \frac{1}{|z_v'(y)|}\right) \, dy + 2\int_a^{b} \frac{1}{|z_v'(y)|} \, dy \\
&< \frac{1}{2}\int_0^{a} \frac{1}{|z_u'(y)|} \, dy + 2\int_0^{a} \frac{1}{|z_v'(y)|} \, dy + 2\int_a^{b} \frac{1}{|z_v'(y)|} \, dy \\
&= \frac{1}{2}\int_0^{a} \frac{1}{|z_u'(y)|} \, dy + 2\int_0^{b} \frac{1}{|z_v'(y)|} \, dy \\
&= \frac{1}{2}\int_0^{c} (u'(x))^2 \, dx + 2\int_0^{d} (v'(x))^2 \, dx \\
&= 2\int_0^{c} (u'(x))^2 \, dx + 2\int_0^{d} (v'(x))^2 \, dx - \frac{3}{2}\int_0^{c} (u'(x))^2 \, dx \\
&= \frac{1}{2}\|u'\|^2 + \frac{1}{2}\|v'\|^2 - \frac{3}{4}\|u'\|^2 \\
&= \frac{1}{2}\|w'\|^2 - \frac{3}{4}\|u'\|^2 \\
&\leq \frac{1}{2}\|w'\|^2 - \frac{3}{4}\min\{\|u'\|^2, \|v'\|^2\}.
\end{aligned}$$

Thus (2.41) is proved in the special case when $u' < 0$ on $(0, c)$ and $v' < 0$ on

$(0, d)$.

Now we consider the general case, which we can reduce to the case treated above as follows.

Let $\phi_1(x)$ be a smooth, even function such that $\phi_1(x) > 0$ for $x \in (0, c)$ and $\phi_1(x) = 0$ for $x \geq c$, and such that $\phi_1(x)$ is strictly decreasing on $(0, c)$. Let $\phi_2(x)$ be a similar function with support on $(0, d)$. For each $\epsilon > 0$, define $u_\epsilon(x) = u(x)+\epsilon\phi_1(x)$ and $v_\epsilon(x) = v(x)+\epsilon\phi_2(x)$, and let $w_\epsilon(x) = u_\epsilon(x)+v_\epsilon(x-T)$. Since $u' \leq 0$ and $\phi_1' < 0$ on $(0, c)$, then $u_\epsilon' = u' + \epsilon\phi_1' < 0$ on $(0, c)$, so u_ϵ is strictly decreasing on $(0, c)$. Similarly, v_ϵ is strictly decreasing on $(0, d)$. So, by what has been proved above,

$$\|(w_\epsilon^*)'\|^2 \leq \|w_\epsilon'\|^2 - \frac{3}{4}\min\{\|u_\epsilon'\|^2, \|v_\epsilon'\|^2\}. \tag{2.45}$$

Now take limits on both sides of (2.45) as ϵ goes to zero. By the dominated convergence theorem, the right hand side approaches

$$\|w'\|^2 - \frac{3}{4}\min\{\|u'\|^2, \|u'\|^2\}.$$

Also, since w_ϵ converges in H^1 norm on $\mathbb{R}$ to w, then by a theorem of Coron [19], w_ϵ^* converges in H^1 norm to w^*. Therefore the left-hand side of (2.41) converges to $\|(w^*)'\|^2$, and (2.41) is proved. □

2.4 Proof of subadditivity

We are now able to prove the following subadditivity property of $I(s, t)$.

Lemma 2.14. *Let $s_1, s_2, t_1, t_2 \geq 0$ be given, and suppose that $s_1 + s_2 > 0$,*

$t_1 + t_2 > 0$, $s_1 + t_1 > 0$, and $s_2 + t_2 > 0$. Then

$$I(s_1 + s_2, t_1 + t_2) < I(s_1, t_1) + I(s_2, t_2). \tag{2.46}$$

Proof. We claim first that, for $i = 1, 2$, we can choose minimizing sequences $(f_n^{(i)}, g_n^{(i)})$ for $I(s_i, t_i)$ such that for all $n \in \mathbb{N}$, $f_n^{(i)}$ and $g_n^{(i)}$

(i) are real-valued and non-negative on $\mathbb{R}$;

(ii) belong to H^1 and have compact support;

(iii) are even functions;

(iv) are non-increasing functions of x for $x \geq 0$;

(v) are C^∞ functions; and

(vi) satisfy $\|f_n^{(i)}\| = s_i$ and $\|g_n^{(i)}\| = t_i$.

To prove this, we can take $i = 1$, since the proof for $i = 2$ is identical. Also we may assume that $s_1 > 0$ and $t_1 > 0$, since otherwise we can simply take $f_n^{(1)}$ or $g_n^{(1)}$ to be identically zero on $\mathbb{R}$.

Start with an arbitrary minimizing sequence $(w_n^{(1)}, z_n^{(1)})$ for $I(s_1, t_1)$. Since functions with compact support are dense in H^1, and $E : Y \to \mathbb{R}$ is continuous, we can approximate $(w_n^{(1)}, z_n^{(1)})$ by functions $(w_n^{(2)}, z_n^{(2)})$ which have compact support and which still form a minimizing sequence for $I(s_1, t_1)$. Then from Lemma 2.12 it follows that the sequence defined by

$$(w_n^{(3)}, z_n^{(3)}) = (|w_n^{(2)}|^*, |z_n^{(2)}|^*)$$

is still a minimizing sequence for $I(s_1, t_1)$, and for each n, $w_n^{(3)}$ and $z_n^{(3)}$have the properties (i) through (iv) listed above.

Next, observe that if f and ψ are any two functions with properties (i) through (iv), then their convolution $f \star \psi$, defined as in (1.13), also satisfies properties (i) through (iv). Moreover, as is well known, if we define $\psi_\epsilon = (1/\epsilon)\psi(x/\epsilon)$ for $\epsilon > 0$, and choose ψ such that $\int_{-\infty}^{\infty} \psi(x)\, dx = 1$, then convolution with ψ_ϵ is an "approximation to the identity": that is, the functions $f \star \psi_\epsilon$ converge strongly to f in H^1 as $\epsilon \to 0$. Finally, if ψ is C^∞ then $f \star \psi_\epsilon$ will be C^∞ also. Therefore by choosing $\psi(x)$ to be any non-negative, C^∞, even function with compact support, which is decreasing for $x \geq 0$, and satisfies $\int_{-\infty}^{\infty} \psi(x)\, dx = 1$, and defining

$$(w_n^{(4)}, z_n^{(4)}) = (w_n^{(3)} \star \psi_{\epsilon_n}, z_n^{(3)} \star \psi_{\epsilon_n}),$$

with ϵ_n chosen appropriately small for n large, we obtain a minimizing sequence $(w_n^{(4)}, z_n^{(4)})$ for $I(s_1, t_1)$ that satisfies not only the properties (i) through (iv) above, but also property (v).

Finally, we obtain the desired minimizing sequence satisfying properties (i) through (vi) by setting

$$f_n^{(1)} = \frac{(s_i)^{1/2} w_n^{(4)}}{\|w_n^{(4)}\|} \quad \text{and} \quad g_n^{(1)} = \frac{(t_i)^{1/2} z_n^{(4)}}{\|g_n^{(i)}\|},$$

respectively, which is possible since for n sufficiently large we have $\|w_n^{(4)}\| > 0$ and $\|z_n^{(4)}\| > 0$.

Next, choose for each n a number x_n such that $f_n^{(1)}(x)$ and $\tilde{f}_n^{(2)}(x) = f_n^{(2)}(x + x_n)$ have disjoint support, and $g_n^{(1)}(x)$ and $\tilde{g}_n^{(2)}(x) = g_n^{(2)}(x + x_n)$ have disjoint support. Define

$$f_n = \left(f_n^{(1)} + \tilde{f}_n^{(2)}\right)^*,$$

$$g_n = \left(g_n^{(1)} + \tilde{g}_n^{(2)}\right)^*.$$

Then $\|f_n\|^2 = s_1 + s_2$ and $\|g_n\|^2 = t_1 + t_2$, so

$$I(s_1 + s_2, t_1 + t_2) \le E(f_n, g_n). \tag{2.47}$$

On the other hand, from Lemma 2.13 we have that

$$\begin{aligned}\int_{-\infty}^{\infty} \left(f_{nx}^2 + g_{nx}^2\right)\, dx &\le \int_{-\infty}^{\infty} \left((f_n^{(1)} + \tilde{f}_n^{(2)})_x^2 + (g_n^{(1)} + \tilde{g}_n^{(2)})_x^2\right)\, dx - K_n \\ &= \int_{-\infty}^{\infty} \left((f_{nx}^{(1)})^2 + (\tilde{f}_{nx}^{(2)})^2 + (g_{nx}^{(1)})^2 + (\tilde{g}_{nx}^{(2)})^2\right)\, dx - K_n,\end{aligned} \tag{2.48}$$

where

$$K_n = \frac{3}{4}\left(\min\left\{\|f_{nx}^{(1)}\|^2, \|f_{nx}^{(2)}\|^2\right\} + \min\left\{\|g_{nx}^{(1)}\|^2, \|g_{nx}^{(2)}\|^2\right\}\right). \tag{2.49}$$

Furthermore, from the properties (2.39), (2.38), and (2.40) of rearrangements, we have that

$$\begin{aligned}\int_{-\infty}^{\infty} |f_n|^{q+2}\, dx &= \int_{-\infty}^{\infty} |f_n^{(1)}|^{q+2}\, dx + \int_{-\infty}^{\infty} |f_n^{(2)}|^{q+2}\, dx \\ \int_{-\infty}^{\infty} g_n^{p+2}\, dx &= \int_{-\infty}^{\infty} (g_n^{(1)})^{p+2}\, dx + \int_{-\infty}^{\infty} (g_n^{(2)})^{q+2}\, dx \\ \int_{-\infty}^{\infty} |f_n|^2 g_n\, dx &\ge \int_{-\infty}^{\infty} |f_n^{(1)}|^2 g_n^{(1)}\, dx + \int_{-\infty}^{\infty} |f_n^{(2)}|^2 g_n^{(2)}\, dx,\end{aligned} \tag{2.50}$$

and therefore, combining with (2.47) and (2.48), we have that for every n,

$$I(s_1 + t_1, s_2 + t_2) \le E(f_n, g_n) \le E(f_n^{(1)}, g_n^{(1)}) + E(f_n^{(2)}, g_n^{(2)}) - K_n. \tag{2.51}$$

It follows by taking the limit superior on the right-hand side that

$$I(s_1 + t_1, s_2 + t_2) \le I(s_1, t_1) + I(s_2, t_2) - \liminf_{n\to\infty} K_n. \tag{2.52}$$

Since $t_1 + t_2 > 0$, then either t_1 and t_2 are both positive, or one of t_1 and t_2

is zero and the other is positive. In the latter case, we may assume that $t_1 = 0$ and $t_2 > 0$, since otherwise we can simply switch t_1 and t_2. Then we will argue separately according as to whether s_2 is positive or zero. To prove the theorem, then, it suffices to consider the following three cases: (i) $t_1 > 0$ and $t_2 > 0$; (ii) $t_1 = 0$, $t_2 > 0$, and $s_2 > 0$; and (iii) $t_1 = 0$, $t_2 > 0$, and $s_2 = 0$.

In case (i), when $t_1 > 0$ and $t_2 > 0$, it follows from Lemma 2.4 that there exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that for all sufficiently large n, $\|(g_n^{(1)})_x\| \geq \delta_1$ and $\|(g_n^{(2)})_x\| \geq \delta_2$. (Note that by Lemma 2.8, this is still true even when $s_1 = 0$ or $s_2 = 0$.) So, letting $\delta = \min(\delta_1, \delta_2) > 0$, (2.49) gives $K_n \geq 3\delta/4$ for all sufficiently large n. From (2.52) we then have that

$$I(s_1 + t_1, s_2 + t_2) \leq I(s_1, t_1) + I(s_2, t_2) - 3\delta/4 < I(s_1, t_1) + I(s_2, t_2), \quad (2.53)$$

as desired.

In case (ii), we have $t_1 = 0$, $t_2 > 0$, $s_2 > 0$, and, since $s_1 + t_1 > 0$ by assumption, $s_1 > 0$ also. By Lemma 2.8 there exists $\delta_1 > 0$ such that for all sufficiently large n, $\|(f_n^{(1)})_x\| \geq \delta_1$.

If, in case (ii), $\beta_1 > 0$, then by Lemma 2.8 there also exists $\delta_2 > 0$ such that for all sufficiently large n, $\|f_{nx}^{(2)}\| \geq \delta_2$. Letting $\delta = \min(\delta_1, \delta_2) > 0$, we get $K_n \geq 3\delta/4$ for large n, and (2.53) follows from (2.52) as in case (i).

On the other hand, if in case (ii) we have $\beta_1 = 0$, then by Lemma 2.8 we have $I(s_1, t_1) = I(s_1, 0) = 0$, and $I(s_1 + s_2, t_1 + t_2) = I(s_1 + s_2, t_2)$ is the infimum of

$$E(f, g) = \int_{-\infty}^{\infty} \left(|f_x|^2 + g_x^2 - \beta_2 g^{p+2} - \alpha |f|^2 g \right) \, dx \quad (2.54)$$

over all $f \in H^1_{\mathbb{C}}$ and $g \in H^1$ such that $\|f\|^2 = s_1 + s_2$ and $\|g\|^2 = t_2$. By Lemma

2.9, there exists $\delta > 0$ such that for all sufficiently large n,

$$\int_{-\infty}^{\infty} \left(|f_{nx}^{(2)}|^2 - \alpha |f_n^{(2)}|^2 g_n^{(2)}\right) \, dx \leq -\delta.$$

Let

$$f_n = \sqrt{\frac{s_1 + s_2}{s_2}} f_n^{(2)}; \tag{2.55}$$

then $\|f_n\|^2 = s_1 + s_2$ and from (2.54) we see that, for all sufficiently large n,

$$\begin{aligned} I(s_1 + s_2, t_2) &\leq E(f_n, g_n^{(2)}) = E(f_n^{(2)}, g_n^{(2)}) + \frac{s_1}{s_2} \int_{-\infty}^{\infty} \left(|f_{nx}^{(2)}|^2 - \alpha |f_n^{(2)}|^2 g_n^{(2)}\right) \, dx \\ &\leq E(f_n^{(2)}, g_n^{(2)}) - \frac{s_1 \delta}{s_2}. \end{aligned} \tag{2.56}$$

This implies, after taking the limit as $n \to \infty$, that

$$I(s_1 + s_2, t_2) \leq I(s_2, t_2) - \frac{s_1 \delta}{s_2} < I(s_2, t_2) = I(s_1, t_1) + I(s_2, t_2), \tag{2.57}$$

as desired. Thus the proof is complete in case (ii).

In case (iii), we have $s_1 > 0$ and $t_2 > 0$, and we have to prove

$$I(s_1, t_2) < I(s_1, 0) + I(0, t_2). \tag{2.58}$$

Let g_0 be as defined in Lemma 2.6 with $t = t_2$, so that $I(0, t_2) = J(g_0)$.

If $\beta_1 > 0$, we have from Lemma 2.7 that $I(s_1, 0) = \tilde{J}(f_0)$, where f_0 is as defined in (2.20) with $s = s_1$. Clearly,

$$\int_{-\infty}^{\infty} |f_0|^2 g_0 \, dx > 0,$$

and so

$$I(s_1,t_2) \le E(f_0,g_0) = \tilde{J}(f_0) + J(g_0) + \int_{-\infty}^{\infty} |f_0|^2 g_0 \; dx \\ < \tilde{J}(f_0) + J(g_0) = I(s_1,0) + I(0,t_2), \tag{2.59}$$

as desired.

On the other hand, if $\beta_1 = 0$, then $I(s_1,0) = 0$ by Lemma 2.8. By Lemma 2.5, there exists $f \in H^1$ such that $\|f\|^2 = s_1$ and

$$\int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g_0\right) \; dx < 0, \tag{2.60}$$

and hence

$$I(s_1,t_1) \le E(f,g_0) = \int_{-\infty}^{\infty} \left(f_x^2 - \alpha f^2 g_0\right) \; dx + J(g_0) < J(g_0), \tag{2.61}$$

which proves (2.58). The proof of Lemma 2.14 is now complete in all cases. □

2.5 Existence of solitary waves

In this section we prove the following existence result.

Theorem 2.15. *Suppose $\alpha > 0$, $\tau_1 \ge 0$, $\tau_2 > 0$, $1 \le q < 4$, and $1 \le p < 4$, where p is a rational number with odd denominator. For $s > 0$ and $t > 0$, define*

$$\mathcal{S}_{s,t} = \left\{ (\phi,\psi) \in Y : E(\phi,\psi) = I(s,t), \int_{-\infty}^{\infty} |\phi|^2 \; dx = s, \text{ and } \int_{-\infty}^{\infty} \psi^2 \; dx = t \right\}. \tag{2.62}$$

Then the following statements are true for all $s > 0$ and $t > 0$.

(i) Every minimizing sequence $\{(f_n,g_n)\}$ for $I(s,t)$ is relatively compact in Y up to translations. That is, there exists a subsequence $\{(f_{n_k},g_{n_k})\}$ and a sequence

of real numbers $\{y_k\}$ *such that* $\{(f_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k))\}$ *converges strongly in* Y *to some* (ϕ, ψ) *in* $\mathcal{S}_{s,t}$*. In particular, the set* $\mathcal{S}_{s,t}$ *is non-empty.*

(ii) Each function $(\phi, \psi) \in \mathcal{S}_{s,t}$ *is a solution of* (1.8) *for some* σ *and* c*, and therefore when substituted into* (1.7) *yields a solitary-wave solution of* (1.3).

(iii) For every (ϕ, ψ) *in* $\mathcal{S}_{s,t}$*, we have that* $\psi(x) > 0$ *for all* $x \in \mathbb{R}$*, and there exist a number* $\theta \in \mathbb{R}$ *and a function* $\tilde{\phi}$ *such that* $\tilde{\phi}(x) > 0$ *for all* $x \in \mathbb{R}$*, and* $\phi(x) = e^{i\theta}\tilde{\phi}(x)$*. Also, the functions* ψ *and* ϕ *are infinitely differentiable on* $\mathbb{R}$.

We begin by establishing the relative compactness, up to translations, of minimizing sequences for $I(s,t)$. Let $\{(f_n, g_n)\}$ be a given minimizing sequence, and define an associated sequence of functions ρ_n by

$$\rho_n = |f_n|^2 + g_n^2.$$

We then have

$$\int_{-\infty}^{\infty} \rho_n(x)\ dx = s + t$$

for all n. The sequence of functions $M_n : [0, \infty) \to [0, s+t]$ defined by

$$M_n(r) = \sup_{y \in \mathbb{R}} \int_{y-r}^{y+r} \rho_n(x)\ dx.$$

is a uniformly bounded sequence of nondecreasing functions on $[0, \infty)$, and therefore (by Helly's selection theorem, for example) has a subsequence, which we will still denote by M_n, that converges pointwise to a nondecreasing function M on $[0, \infty)$. Then

$$\gamma = \lim_{r \to \infty} M(r) \tag{2.63}$$

exists and satisfies $0 \leq \gamma \leq s + t$.

From Lions' Concentration Compactness Lemma, Theorem 2.1 above, there are three possibilities for the value of γ :

(a) Case 1 : (*Vanishing*) $\gamma = 0$. Since $M(r)$ is non-negative and nondecreasing, this is equivalent to saying

$$M(r) = \lim_{n\to\infty} M_n(r) = \lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} \rho_n(x)\ dx = 0,$$

for all $r < \infty$, or

(b) Case 2 : (*Dichotomy*) $\gamma \in (0, s+t)$, or

(c) Case 3 : (*Compactness*) $\gamma = s+t$, that is, there exists $\{y_n\} \subset \mathbb{R}$ such that $\rho_n(.+y_n)$ is tight, namely, for all $\epsilon > 0$, there exists $r < \infty$ such that

$$\int_{y-r}^{y+r} \rho_n(x)\ dx \geq (s+t) - \epsilon.$$

We claim now that $\gamma > 0$. To prove this, we require the following lemma.

Lemma 2.16. *Suppose w_n is a sequence of functions which is bounded in H^1 and which satisfies, for some $R > 0$,*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} w_n^2\ dx = 0. \tag{2.64}$$

Then for every $r > 2$,

$$\lim_{n\to\infty} |w_n|_r = 0.$$

Proof. This is a special case of Lemma I.1 of part 2 of [33], but for the sake of completeness we give a proof here. Let

$$\epsilon_n = \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} w_n^2\ dx, \tag{2.65}$$

so that $\lim_{n\to\infty} \epsilon_n = 0$. For every $y \in \mathbb{R}$, we have by standard Sobolev inequalities (see Theorem 10.1 of [24]) that

$$\int_{y-R}^{y+R} |w_n|^r \ dx \leq C \left(\int_{y-R}^{y+R} |w_n|^2 \ dx \right)^s \left(\int_{y-R}^{y+R} \left(w_n^2 + w_{nx}^2\right) \ dx \right)^{1+s},$$

where $s = (r-2)/4$. It then follows from (2.64) that

$$\begin{aligned} \int_{y-R}^{y+R} |w_n|^r \ dx &\leq C\epsilon_n^s \left(\int_{y-R}^{y+R} \left(w_n^2 + w_{nx}^2\right) \ dx \right) \|w_n\|_1^s \\ &\leq C\epsilon^s \int_{y-R}^{y+R} \left(w_n^2 + w_{nx}^2\right) \ dx. \end{aligned} \tag{2.66}$$

Now if we cover $\mathbb{R}$ by intervals of length R in such a way that each point of $\mathbb{R}$ is contained in at most two of the intervals, then by summing (2.66) over all the intervals in the cover, we obtain that

$$|w_n|_r \leq 3C\epsilon_n^s \|w_n\|_1^2 \leq C\epsilon_n^s,$$

from which the desired result follows. $\square$

Next we prove that

$$\gamma \neq 0. \tag{2.67}$$

Indeed, suppose for the sake of contradiction that $\gamma = 0$. Then (2.64) holds both for $w_n = |f_n|$ and for $w_n = g_n$. Since both $\{|f_n|\}$ and $\{g_n\}$ are bounded sequences in H^1 by Lemma 2.3, then Lemma (2.16) implies that for every $r > 2$, f_n and g_n converge to 0 in L^r norm. Since

$$\left| \int_{-\infty}^{\infty} |f_n|^2 g_n \ dx \right| \leq |f_n|_4^{1/2} \|g_n\|$$

and $\|g_n\|$ is bounded, it follows also that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}|f_n|^2 g_n\ dx = 0.$$

Hence

$$I(s,t) = \lim_{n\to\infty} E(f_n, g_n) \geq \liminf_{n\to\infty}\int_{-\infty}^{\infty}\left(|f_{nx}|^2 + g_{nx}^2\right) dx \geq 0, \tag{2.68}$$

contradicting Lemma 2.3. This proves (2.67).

Lemma 2.17. *Suppose γ is defined as in* (2.63). *Then there exist numbers $s_1 \in [0, s]$ and $t_1 \in [0, t]$ such that*

$$\gamma = s_1 + t_1 \tag{2.69}$$

and

$$I(s_1, t_1) + I(s - s_1, t - t_1) \leq I(s, t). \tag{2.70}$$

Proof. Since the proof is almost the same as the proof of Lemma 3.10 of [3], with only slight modifications, we just give an outline here, and refer to [3] for the details. Let ρ and σ be smooth functions on $\mathbb{R}$ such that $\rho^2+\sigma^2 = 1$ on $\mathbb{R}$, and ρ is identically 1 on $[-1, 1]$ and is supported in $[-2, 2]$; and define $\rho_\omega(x) = \rho(x/\omega)$ and $\sigma_\omega(x) = \sigma(x/\omega)$ for $\omega > 0$. From the definition of γ it follows that for given $\epsilon > 0$, there exist $\omega > 0$ and a sequence y_n such that, after passing to a subsequence, the functions $(f_n^{(1)}(x), g_n^{(1)}(x)) = \rho_\omega(x - y_n)(f_n(x), g_n(x))$ and $(f_n^{(2)}(x), g_n^{(2)}(x)) = \sigma_\omega(x - y_n)(f_n(x), g_n(x))$ satisfy $\|f_n^{(1)}\|^2 \to s_1$, $\|g_n^{(1)}\|^2 \to t_1$, $\|f_n^{(2)}\|^2 \to s - s_1$, and $\|g_n^{(2)}\|^2 \to t - t_1$ as $n \to \infty$, where $|(s_1 + t_1) - \alpha| < \epsilon$, and

$$E(f_n^{(1)}, g_n^{(1)}) + E(f_n^{(2)}, g_n^{(2)}) \leq E(f_n, g_n) + C\epsilon \tag{2.71}$$

for all n. To prove (2.71), one writes

$$E(f_n^{(1)}, g_n^{(1)}) = \int_{-\infty}^{\infty} \rho_\omega^2 \left(|f_{nx}|^2 + g_{nx}^2 - \beta_1|f_n|^{q+2} - \beta_2 g_n^{p+2} - \alpha|f_n|^2 g_n\right)\ dx$$
$$+ \int_{-\infty}^{\infty} \left((\rho_\omega^2 - \rho_\omega^{q+2})\beta_1|f_n|^{q+2} + (\rho_\omega^2 - \rho_\omega^{p+2})\beta_2|g_n|^{p+2} + (\rho_\omega^2 - \rho_\omega^3)\alpha|f_n|^2 g_n\right)\ dx$$
$$\int_{-\infty}^{\infty} \left((\rho_\omega')^2(|f_{nx}|^2 + g_{nx}^2) + 2\rho_\omega\rho_\omega'(\mathrm{Re}\ f_n(\overline{f_n})_x + g_n g_{nx}) + (\rho_\omega')^2|f|^2\right)\ dx,$$

and observes that the last two integrals on the right hand side can be made arbitrarily uniformly small by taking ω sufficiently large. A similar estimate obtains for $E(f_n^{(2)}, g_n^{(2)})$, and (2.71) follows by adding the two estimates and using $\rho_\omega^2 + \sigma_\omega^2 = 1$.

Now we show that the limit inferior as $n \to \infty$ of the left-hand side of (2.71) is greater than or equal to $I(s_1, t_1) + I(s - s_1, t - t_1)$. If s_1, t_1, $s - s_1$, and $t - t_1$ are all positive, this follows by rescaling $f_n^{(i)}$ and $g_n^{(i)}$ for $i = 1, 2$ so that $\|f_n^{(1)}\|^2 = s_1$, $\|g_n^{(1)}\|^2 = t_1$, $\|f_n^{(2)}\|^2 = s - s_1$, and $\|g_n^{(2)}\|^2 = t - t_1$, since the scaling factors tend to 1 as $n \to \infty$. On the other hand, if $s_1 = 0$ and $t_1 > 0$ then as in (2.68) we have

$$\lim_{n\to\infty} E(f_n^{(1)}, g_n^{(1)}) = \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|f_{nx}^{(1)}|^2 + (g_{nx}^{(1)})^2 - \beta_2(g_n^{(1)})^{q+2}\right)\ dx$$
$$\geq \liminf_{n\to\infty} \int_{-\infty}^{\infty} ((g_{nx}^{(1)})^2 - \beta_2(g_n^{(1)})^{q+2})\ dx \geq I(0, t_1),$$

and similar estimates hold if t_1, $s - s_1$, or $t - t_1$ are zero.

Taking then the limit inferior of the left-hand side and the limit of the right-hand side of (2.71) as $n \to \infty$, we obtain

$$I(s_1, t_1) + I(s - s_1, t - t_1) \leq I(s, t) + C\epsilon,$$

which proves (2.70), as ϵ is arbitrary. $\square$

The next lemma shows that the dichotomy alternative of Lions' Concentration Compactness Lemma does not hold here.

Lemma 2.18. *Suppose* $s, t > 0$, *and let* $\{(f_n, g_n)\}$ *be any minimizing sequence for* $I(s,t)$. *Then*

$$\gamma = s + t. \tag{2.72}$$

Proof. Suppose to the contrary that $\gamma < s + t$. Let s_1 and t_1 be as defined in Lemma 2.17, and let $s_2 = s - s_1$ and $t_2 = t - t_1$. Then $s_2 + t_2 = (s + t) - \gamma > 0$, and also (2.67) and (2.69) imply that $s_1 + t_1 > 0$. Moreover, $s_1 + s_2 = s > 0$ and $t_1 + t_2 = t > 0$. Therefore Lemma 2.14 implies that that (2.46) holds. But this contradicts (2.70). Thus (2.72) is proved. $\square$

Once we have ruled out both vanishing and dichotomy, assertion (i) of Theorem 2.15, concerning the relative compactness of minimizing sequences up to translation, can be proved. Indeed, Lions' Concentration Compactness Lemma guarantees that sequence $\{\rho_n\}$ is tight, i.e. there exists a sequence of real numbers $\{y_n\}$ such that for every $k \in \mathbb{N}$, there exists $\omega_k \in \mathbb{R}$ such that, for all sufficiently large n,

$$\int_{y_n-\omega_k}^{y_n+\omega_k} \left(|f_n|^2 + g_n^2\right)\, dx > s + t - \frac{1}{k}. \tag{2.73}$$

Let us now define $w_n(x) = f_n(x + y_n)$ and $z_n(x) = g_n(x + y_n)$. Then, by (2.73), for every $k \in \mathbb{N}$, we have

$$\int_{-\omega_k}^{\omega_k} \left(|w_n|^2 + z_n^2\right)\, dx > s + t - \frac{1}{k}, \tag{2.74}$$

for all sufficiently large n. (In other words, the measures

$$\mu_n = (|w_n|^2 + z_n)\ dx$$

form a "tight" family on $\mathbb{R}$, in the sense that for every $\epsilon > 0$, there exists a fixed compact set K such that $\mu_n(\mathbb{R}\backslash K) < \epsilon$ for all $n \in \mathbb{N}$.) Since $\{(w_n, z_n)\}$ is bounded uniformly in Y, there exists a subsequence, denoted again by $\{(w_n, z_n)\}$, which converges weakly in Y to a limit $(\phi, \psi) \in Y$. Then Fatou's lemma implies that

$$\|\phi\|^2 + \|\psi\|^2 \leq \liminf_{n\to\infty} \int_{-\infty}^{\infty} \left(|w_n|^2 + z_n^2\right)\ dx = s + t.$$

Moreover, for fixed k, (w_n, z_n) converges weakly in $H^1(-\omega_k, \omega_k) \times H^1(-\omega_k, \omega_k)$ to (ϕ, ψ), and therefore has a subsequence, denoted again by $\{(w_n, z_n)\}$, which converges strongly to (ϕ, ψ) in $L^2(-\omega_k, \omega_k) \times L^2(-\omega_k, \omega_k)$. By a diagonalization argument, we may assume that the subsequence has this property for every k simultaneously. It then follows from (2.74) that

$$\int_{-\infty}^{\infty} \left(|\phi|^2 + \psi^2\right)\ dx \geq \int_{-\omega_k}^{\omega_k} \left(|\phi|^2 + \psi^2\right)\ dx \geq s + t - \frac{1}{k}.$$

Since k was arbitrary, we get

$$\int_{-\infty}^{\infty} \left(|\phi|^2 + \psi^2\right)\ dx = s + t,$$

which implies that (w_n, z_n) converges strongly to the limit (ϕ, ψ) in $L^2 \times L^2$.

Next, observe that

$$\int_{-\infty}^{\infty} (z_n |w_n|^2 - \psi |\phi|^2)\ dx = \int_{-\infty}^{\infty} z_n \left(|w_n|^2 - |\phi|^2\right)\ dx + \int_{-\infty}^{\infty} (z_n - \psi) |\phi|^2\ dx. \tag{2.75}$$

For the first integral, we have

$$\left| \int_{-\infty}^{\infty} z_n \left(|w_n|^2 - |\phi|^2 \right) \, dx \right| \leq \|z_n\| \, \|w_n - \phi\| \left(\|w_n\|_1 + \|\phi\|_1 \right)$$

and the right-hand side goes to zero as $n \to \infty$, since $\{(\omega_n, z_n)\}$ is bounded in Y. Similarly, the second integral on the right-hand side of (2.75) converges to zero. It follows then from (2.73) that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} z_n \, |w_n|^2 \, dx = \int_{-\infty}^{\infty} \psi \, |\phi|^2 \, dx.$$

Moreover,

$$|z_n - \psi|_{p+2} \leq C \, \|z_n - \psi\|_1^{1/(p+2)} \, \|z_n - \psi\|^{\frac{p+1}{p+2}} \leq C \, \|z_n - \psi\|^{\frac{p+1}{p+2}},$$

which implies $|z_n|_{p+2}^{p+2} \to |\psi|_{p+2}^{p+2}$ as $n \to \infty$. Also,

$$|w_n - \phi|_{q+2} \leq C \, \|w_n - \phi\|_1^{1/(q+2)} \, \|w_n - \phi\|^{\frac{q+1}{q+2}} \leq C \, \|w_n - \phi\|^{\frac{q+1}{q+2}},$$

and hence $|w_n|_{q+2}^{q+2} \to |\phi|_{q+2}^{q+2}$ as $n \to \infty$. Therefore, by another application of Fatou's lemma, we get

$$I(s,t) = \lim_{n\to\infty} E(w_n, z_n) \geq E(\phi, \psi), \tag{2.76}$$

whence $E(f,g) = I(s,t)$. Thus $(\phi, \psi) \in \mathcal{S}_{s,t}$. Finally, since equality holds in (2.76), then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|w_n'|^2 + (z_n')^2 \right) \, dx = \int_{-\infty}^{\infty} \left(|\phi'|^2 + (\psi')^2 \right) \, dx,$$

so (w_n, z_n) converges strongly to (ϕ, ψ) in the norm of Y. This proves statement

(i) of Theorem 2.15.

Since (ϕ, ψ) is in the minimizing set $\mathcal{S}_{s,t}$ for $I(s,t)$, and so minimizes $E(u,v)$ subject to $H(u)$ and $H(v)$ being held constant, the Lagrange multiplier principle (see, for example, Theorem 7.7.2 of [34]) asserts that there exist real numbers σ and c such that

$$\delta E(\phi, \psi) = \sigma \delta H(\phi) + c \delta H(\psi), \tag{2.77}$$

where δ denotes the Fréchet derivative. Computing the Fréchet derivatives we see that this means that equations (1.8) hold, at least in the sense of distributions. But since the right-hand sides of the equations in (1.8) are continuous functions of the unknowns, distributional solutions are also classical solutions (cf. Lemma 1.3 of [44]). This then proves assertion (ii) of Theorem 2.15.

It remains to prove the assertions in part (iii) of Theorem 2.15.

Multiplying the first equation in (1.8) by $\overline{\phi}$ and integrating over $\mathbb{R}$, we have after an integration by parts that

$$\int_{-\infty}^{\infty} \left(|\phi'|^2 - \tau_1 |\phi|^{q+2} - \alpha |\phi|^2 \psi \right) \, dx = -\sigma \int_{-\infty}^{\infty} |\phi|^2 \, dx = -\sigma s. \tag{2.78}$$

In particular, it follows from (2.78) that σ is real. Similarly, multiplying the second equation in (1.8) by ψ and integrating over $\mathbb{R}$ yields

$$\int_{-\infty}^{\infty} \left(|\psi'|^2 - \frac{\tau_2}{p+1} \psi^{p+2} - \frac{\alpha}{2} |\phi|^2 \psi \right) \, dx = -c \int_{-\infty}^{\infty} |\psi|^2 \, dx = -ct. \tag{2.79}$$

From Lemma 2.9, applied to the constant sequence $(f_n, g_n) = (\phi, \psi)$, we have that

$$\int_{-\infty}^{\infty} \left(|\phi'|^2 - \tau_1 |\phi|^{q+2} - \alpha |\phi|^2 \psi \right) \, dx < 0, \tag{2.80}$$

and since $\tau_1 = \beta_1 (q+2)/2 > \beta_1$, it follows that the integral on the left-hand side of (2.78) is negative, and so we must have $\sigma > 0$. Therefore, a calculation with the Fourier transform shows that the operator $-\partial_x^2 + \sigma$ appearing in the first equation of (1.8) is invertible on $H^1_{\mathbb{C}}$, with inverse given by convolution with the function

$$K_\sigma(x) = \frac{1}{2\sqrt{\sigma}} e^{-\sqrt{\sigma}|x|}.$$

The first equation of (1.8) can then be rewritten in the form

$$\phi = K_\sigma \star \left(\tau_1 |\phi|^q \phi + \alpha \phi \psi\right), \tag{2.81}$$

where $\star$ denotes convolution as in (1.13).

Now we observe that it follows from the first equation in (1.8) that there exist $\theta \in \mathbb{R}$ and a real-valued function $\tilde{\phi}(x)$ such that $\phi(x) = e^{i\theta}\tilde{\phi}(x)$ on $\mathbb{R}$. This is proved for the case $\tau_1 = 0$ in part (iii) of Theorem 2.1 of [3], and it is easy to check that the same proof works as well when $\tau_1 \neq 0$.

Note next that $(\tilde{\phi}, |\psi|)$ and $(|\tilde{\phi}|, |\psi|)$ are also in $\mathcal{S}_{s,t}$, as follows from Lemma (2.10). Therefore, if we let $w = |\tilde{\phi}|$, then $\tilde{\phi}$ and w satisfy the Lagrange multiplier equations

$$\begin{aligned} -\tilde{\phi}'' + \sigma \tilde{\phi} &= \tau_1 w^q \tilde{\phi} + \alpha \tilde{\phi} |\psi| \\ -w'' + \sigma w &= \tau_1 w^q w + \alpha w |\psi|. \end{aligned} \tag{2.82}$$

(That the Lagrange multiplier σ is the same in both equations follows from the fact that σ is determined by the equation (2.78), and this equation is unchanged when ϕ is replaced by w.) Multiplying the first equation by w and the second equation by $\tilde{\phi}$, and subtracting the two equations, we find that the $w\tilde{\phi}'' - \tilde{\phi}w'' = 0$. Therefore the Wronskian $w\tilde{\phi}' - \tilde{\phi}w'$ of w and $\tilde{\phi}$ is constant, and since w and $\tilde{\phi}$ are both in H^1, this constant must be zero. So w and $\tilde{\phi}$ are constant multiples of each other, and hence $\tilde{\phi}$, like w, must be of one sign on $\mathbb{R}$. By replacing θ by

$\theta + i\pi$ if necessary, we can assume that $\tilde{\phi} \geq 0$ on $\mathbb{R}$.

We claim that

$$\int_{-\infty}^{\infty} |\phi|^2 |\psi| \ dx = \int_{-\infty}^{\infty} |\phi|^2 \psi \ dx. \tag{2.83}$$

To prove this, note that since $E(|\phi|, |\psi|) = E(|\phi|, \psi) = I(s,t)$, we have

$$\alpha \int_{-\infty}^{\infty} |\phi|^2 (|\psi| - \psi) \ dx = \int_{-\infty}^{\infty} \left((|\psi_x|^2 - \psi_x^2) - \beta_2 (|\psi|^{p+2} - \psi^{p+2}) \right) \ dx. \tag{2.84}$$

Using (2.36), we see that the right-hand side of this equation is less than or equal to zero, so we must have

$$\alpha \int_{-\infty}^{\infty} |\phi|^2 (|\psi| - \psi) \ dx \leq 0 \tag{2.85}$$

also. But since the integrand is non-negative, this proves (2.83).

From (2.83) it follows that $\psi(x) \geq 0$ at every point x in $\mathbb{R}$ for which $\tilde{\phi}(x) \neq 0$. Now (2.81) implies that

$$\tilde{\phi} = K_\sigma \star \left(\tau_1 |\tilde{\phi}|^q \tilde{\phi} + \alpha \tilde{\phi} \psi \right). \tag{2.86}$$

Since the convolution of K_σ with a function that is everywhere non-negative and not identically zero must produce an everywhere positive function, it follows that $\tilde{\phi}(x) > 0$ for all $x \in \mathbb{R}$. But this in turn implies that $\psi(x) \geq 0$ for all $x \in \mathbb{R}$.

Now suppose, for the sake of contradiction, that $\psi(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Then from the preceding paragraph it follows that x_0 is a point where ψ takes its minimum value over $\mathbb{R}$, and therefore we must have $\psi'(x_0) = 0$. But then standard uniqueness theory for ordinary differential equations, applied to the second equation in (1.8) viewed as an inhomogeneous equation for ψ, yields that ψ must be identically zero on its entire interval of existence about x_0, which in this case is $\mathbb{R}$. But this contradicts the fact that $\|\psi\|^2 = t > 0$. Therefore ψ must be everywhere positive on $\mathbb{R}$.

Finally, since ψ and $|\phi|$ are everywhere positive on $\mathbb{R}$, and the right-hand sides of the equations in (1.8) are infinitely differentiable functions of ϕ and ψ on the domain $\{(\phi, \psi) \in \mathbb{C} \times \mathbb{R} : |\phi| > 0 \text{ and } \psi > 0\}$, it follows from the standard theory of ordinary differential equations that any solution of (1.8) must be infinitely differentiable on its interval of existence.

This completes the proof of Theorem 2.15.

2.6 Stability of solitary waves

In this section we consider the full variational characterization of solitary-wave solutions for (1.3), namely, the problem of finding

$$W(s,t) = \inf\{E(h,g) : (h,g) \in Y,\ H(h) = s \text{ and } G(h,g) = t\}.$$

for any $s > 0$ and $t \in \mathbb{R}$. Following our usual convention, we define a minimizing sequence for $W(s,t)$ to be a sequence (h_n, g_n) in Y such that

$$\lim_{n\longrightarrow\infty} H(h_n) = s,\ \lim_{n\longrightarrow\infty} G(h_n, g_n) = t \text{ and } \lim_{n\longrightarrow\infty} E(h_n, g_n) = W(s,t).$$

Lemma 2.19. *Suppose $1 \leq q < 4$ and $1 \leq p < 4/3$, and let $s > 0$ and $t \in \mathbb{R}$. If $\{(h_n, g_n)\}$ is a minimizing sequence for $W(s,t)$, then $\{(h_n, g_n)\}$ is bounded in Y.*

Proof. Since $\|h_n\|^2 = H(h_n)$ is bounded, then

$$\begin{aligned} \|g_n\|^2 = \left| G(h_n, g_n) - \operatorname{Im} \int_{-\infty}^{\infty} h_n (\overline{h_n})_x \, dx \right| &\leq C\left(1 + \|h_n\| \cdot \|h_{nx}\|\right) \\ &\leq C\left(1 + \|(h_n, g_n)\|_Y\right), \end{aligned} \tag{2.87}$$

where C is independent of n. Therefore

$$\|(h_n, g_n)\|_Y^2 = E(h_n, g_n) + \int_{-\infty}^{\infty} \left(\beta_1 |h_n|^{q+2} + \beta_2 g_n^{p+2} + \alpha |h_n|^2 g_n\right)\ dx + \|h_n\|^2 + \|g_n\|^2$$
$$\leq C \int_{-\infty}^{\infty} \left(|h_n|^{q+2} + |g_n|^{p+2} + |h_n|^2 |g_n|\right)\ dx + C\left(1 + \|(h_n, g_n)\|_Y\right). \tag{2.88}$$

From (2.87) it follows that

$$\int_{-\infty}^{\infty} |g_n|^{p+2}\ dx \leq C \|g_{nx}\|^{p/2} \|g_n\|^{(p+4)/2}$$
$$\leq C \left(\|(h_n, g_n)\|_Y^{p/2} + \|(h_n, g_n)\|_Y^{(3p+4)/4}\right).$$

On the other hand, as in (2.4), we have

$$\int_{-\infty}^{\infty} |h_n|^{q+2}\ dx \leq C \|h_{nx}\|^{q/2} \|h_n\|^{(q+4)/2} \leq C \|(h_n, g_n)\|_Y^{q/2},$$

and, as in (2.6),

$$\int_{-\infty}^{\infty} |h_n|^2 |g_n|\ dx \leq C \|h_{nx}\|^{1/2} \|g_n\| \leq C\left(1 + \|(h_n, g_n)\|_Y\right).$$

Combining these estimates with (2.88) gives

$$\|(h_n, g_n)\|_Y^2$$
$$\leq C\left(1 + \|(h_n, g_n)\|_Y + \|(h_n, g_n)\|_Y^{q/2} + \|(h_n, g_n)\|_Y^{p/2} + \|(h_n, g_n)\|_Y^{(3p+4)/4}\right),$$

and since $q < 4$ and $p < 4/3$, the exponents on the right-hand side are all less than 2. Hence $\|(h_n, g_n)\|_Y$ is bounded. □

Lemma 2.20. *Suppose $k, \theta \in \mathbb{R}$ and $h \in H^1_{\mathbb{C}}$. If $f(x) = e^{i(kx+\theta)} h(x)$, then*

$$E(f, g) = E(h, g) + k^2 H(h) - 2k\ \mathrm{Im} \int_{-\infty}^{\infty} h \overline{h}_x\ dx$$

and

$$G(f,g) = G(h,g) - kH(h).$$

We omit the proof, which is elementary.

The next lemma gives a relationship between $W(s,t)$ and $I(s,t)$.

Lemma 2.21. *Suppose $s > 0$ and $t \in \mathbb{R}$, and define $b = b(a) = (t-a)/s$ for $a \geq 0$. Then*

$$W(s,t) = \inf_{a \geq 0}\{I(s,a) + b(a)^2 s\}.$$

Proof. First, suppose $a \geq 0$ and let $(h,g) \in Y$ be given such that $\|h\|^2 = s$ and $\|g\|^2 = a$. Let $b = b(a)$ and

$$c = \operatorname{Im} \int_{-\infty}^{\infty} h\overline{h_x}\, dx,$$

and put $f(x) = e^{ikx}h(x)$ with $k = (c/s) - b$. Then from Lemma 2.20,

$$\begin{aligned}
E(f,g) &= E(h,g) + k^2 H(h) - 2k \ \operatorname{Im} \int_{-\infty}^{\infty} h\overline{h_x}\, dx \\
&= E(h,g) + \left(\frac{c}{s} - b\right)^2 s - 2\left(\frac{c}{s} - b\right) c \\
&= E(h,g) + b^2 s - \frac{c^2}{s} \leq E(h,g) + b^2 s, \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
G(f,g) &= G(h,g) - kH(h) \\
&= \|g\|^2 + \operatorname{Im} \int_{-\infty}^{\infty} h\overline{h_x}\, dx - \left(\frac{c}{s} - b\right) H(h) \\
&= a + c - \left(\frac{c}{s} - b\right) s = a + bs = t.
\end{aligned}$$

Since $H(f) = s$, we conclude that

$$W(s,t) \leq E(f,g) \leq E(h,g) + b(a)^2 s.$$

Taking the infimum over the set of functions (h, g) such that $H(h) = s$ and $\|g\|^2 = a$ gives

$$W(s,t) \leq I(s,a) + b(a)^2 s,$$

and taking the infimum over a gives

$$W(s,t) \leq \inf_{a \geq 0} \left\{ I(s,a) + b(a)^2 s \right\}.$$

To prove the reverse inequality, let $s > 0$ and $t \in \mathbb{R}$ be given. Suppose that $(h, g) \in Y$ is given such $H(h) = s$ and $G(h, g) = t$. We will show that there exists $a \geq 0$ such that

$$E(h,g) \geq I(s,a) + b(a)^2 s.$$

Choose $a = \|g\|^2$. Then

$$a = t - \operatorname{Im} \int_{-\infty}^{\infty} h \overline{h_x} \, dx.$$

Define $f(x) = e^{ibx} h(x)$, where $b = b(a) = (t - a)/s$. Then

$$\begin{aligned} E(e^{ibx} h, g) &= E(h,g) + b^2 H(h) - 2b \ \operatorname{Im} \int_{-\infty}^{\infty} h \overline{h_x} \, dx \\ &= E(h,g) + b^2 s - 2b(t-a) = E(h,g) - b^2 s. \end{aligned}$$

Since $\|f\|^2 = \|h\|^2 = s$ and $\|g\|^2 = a$, we have $a \geq 0$ and $I(s,a) \leq E(f,g)$. Hence

$$\begin{aligned} E(h,g) &= E(f,g) + b^2 s \geq I(s,a) + b^2 s \\ &\geq \inf_{a \geq 0} \left\{ I(s,a) + b(a)^2 s \right\}. \end{aligned}$$

Taking infimum over h and g such that $H(h) = \|h\|^2 = s$ and $G(h,g) = t$ gives

$$W(s,t) \geq \inf_{a \geq 0} \left\{ I(s,a) + b(a)^2 s \right\}.$$

Combining both inequalities, we get the desired conclusion. $\square$

Lemma 2.22. *Suppose $s > 0$ and $t \in \mathbb{R}$, and define $b(a) = (t-a)/s$ for $a \geq 0$. If $\{(h_n, g_n)\}$ is a minimizing sequence for $W(s,t)$, then there exists a subsequence (still denoted by $\{(h_n, g_n)\}$) and a number $a \geq 0$ such that*

$$\lim_{n\to\infty} \|g_n\|^2 = a,$$

$$\lim_{n\to\infty} E(e^{ib(a)x} h_n, g_n) = I(s,a),$$

and

$$W(s,t) = I(s,a) + b(a)^2 s. \tag{2.89}$$

If $\beta_1 = 0$, we can further assert that $a > 0$.

Proof. The sequence a_n defined by

$$a_n = \|g_n\|^2 = G(h_n, g_n) - \operatorname{Im} \int_{-\infty}^{\infty} h_n \overline{h_{nx}}\, dx = t - \operatorname{Im} \int_{-\infty}^{\infty} h_n \overline{h_{nx}}\, dx$$

is bounded, by Lemma 2.19. Hence, by passing to a subsequence, we may assume that a_n converges to a limit $a \geq 0$. Let $b = b(a)$ and define $f_n(x) = e^{ibx} h_n(x)$. Then from Lemmas 2.20 and 2.21 we have that

$$\begin{aligned} \lim_{n\to\infty} E(f_n, g_n) &= \lim_{n\to\infty} \left(E(h_n, g_n) + b^2 H(h_n) - 2b \operatorname{Im} \int_{-\infty}^{\infty} h_n \overline{h_{nx}}\, dx \right) \\ &= W(s,t) + b^2 s - 2b(t-a) = W(s,t) - b^2 s \leq I(s,a). \end{aligned} \tag{2.90}$$

We claim that also

$$\lim_{n\to\infty} E(f_n, g_n) \geq I(s, a). \tag{2.91}$$

For if $a > 0$, then for sufficiently large n we have that $\|f_n\| > 0$ and $\|g_n\| > 0$, so the sequences $\beta_n = \sqrt{s}/\|f_n\|$ and $\theta_n = \sqrt{a}/\|g_n\|$ are defined, and both approach 1 as $n \to \infty$. Since $\|\beta_n f_n\|^2 = s$ and $\|\theta_n g_n\|^2 = a$, then $E(\beta_n f_n, \theta_n g_n) \geq I(s, a)$, and therefore

$$\lim_{n\to\infty} E(f_n, g_n) = \lim_{n\to\infty} E(\beta_n f_n, \theta_n g_n) \geq I(s, a).$$

On the other hand, if $a = 0$, then $\|g_n\| \to 0$ as $n \to \infty$, so it follows as in the proof of Lemma 2.4 that (2.8) holds: that is,

$$\lim_{n\to\infty} E(f_n, g_n) = \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|f_{nx}|^2 - \beta_1 |f_n|^{q+2}\right)\ dx \geq I(s, 0). \tag{2.92}$$

Hence (2.91) holds in either case.

All the assertions of the Lemma, except the last one, now follow from (2.90) and (2.91).

To prove the last assertion of the Lemma, assume to the contrary that $\beta_1 = 0$ and $a = 0$. From Lemma 2.8 we know that $I(s, a) = 0$, so from (2.89) it follows that $W(s, t) \geq 0$. But on the other hand, we can let g_0 be the function defined in Lemma 2.6, and f_0 be the corresponding function defined for this g_0 in Lemma 2.5. Then f_0 is real, $\|f_0\|^2 = s$, and $\|g_0\|^2 = t$, so $H(f_0) = s$ and $G(f_0, g_0) = t$, and hence $W(s, t) \leq E(f_0, g_0)$. Since

$$E(f_0, g_0) = \int_{-\infty}^{\infty} \left(f_{0x}^2 - \alpha f_0^2 g_0\right)\ dx + J(g_0) < 0,$$

it follows that $W(s, t) < 0$, giving the desired contradiction. $\square$

The following is our stability result:

Theorem 2.23. *Suppose $\alpha > 0$, $\tau_1 \geq 0$, $\tau_2 > 0$, $1 \leq q < 4$, and $p = 1$. For $s > 0$ and $t \in \mathbb{R}$, define*

$$\mathcal{F}_{s,t} = \{(\Phi, \psi) \in Y : E(\Phi, \psi) = W(s,t), H(\Phi) = s, \text{ and } G(\Phi, \psi) = t\}. \quad (2.93)$$

Then the following statements are true for all $s > 0$ and $t \in \mathbb{R}$.

(i) Every minimizing sequence $\{(h_n, g_n)\}$ for $W(s,t)$ is relatively compact in Y up to translations. That is, if

$$\lim_{n\to\infty} H(h_n) = s, \ \lim_{n\to\infty} G(h_n, g_n) = t, \ \text{and} \ \lim_{n\to\infty} E(h_n, g_n) = W(s,t),$$

then there is a subsequence $\{(h_{n_k}, g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that $\{h_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k)\}$ converges strongly in Y to some $(\Phi, \psi) \in \mathcal{F}_{s,t}$. In particular, the set $\mathcal{F}_{s,t}$ is non-empty.

(ii) Each $(\Phi, \psi) \in \mathcal{F}_{s,t}$ is a solution of (1.11) for some ω and c, and therefore when substituted into (1.12) yields a solitary-wave solution of (1.3).

(iii) For every $(\Phi, \psi) \in \mathcal{F}_{s,t}$, let $a = \|\psi\|^2$ and $b = (t - a)/s$. Then there exist $\theta \in \mathbb{R}$ and a real-valued function $\tilde{\phi}$ such that $(\tilde{\phi}, \psi) \in \mathcal{S}_{s,a}$ and

$$\Phi(x) = e^{i(-bx+\theta)}\tilde{\phi}(x) \quad (2.94)$$

on $\mathbb{R}$. Further, if $\tau_1 = 0$, then $a > 0$, $\psi(x) > 0$ for all $x \in \mathbb{R}$, and we can take $\tilde{\phi}$ to be everywhere positive on $\mathbb{R}$.

(iv) The set $\mathcal{F}_{s,t}$ is a stable set of initial data for (1.3), in the following sense:

for every $\epsilon > 0$, there exists $\delta > 0$ such that if $(h_0, g_0) \in Y$,

$$\inf_{(\Phi,\psi)\in\mathcal{F}_{s,t}} \|(h_0, g_0) - (\Phi, \psi)\|_Y < \delta,$$

and $(u(x,t), v(x,t))$ is the solution of (1.3) with

$$(u(x,0), v(x,0)) = (h_0(x), g_0(x)),$$

then for all $t \geq 0$,

$$\inf_{(\Phi,\psi)\in\mathcal{F}_{s,t}} \|(u(\cdot,t), v(\cdot,t)) - (\Phi, \psi)\|_Y < \epsilon.$$

Furthermore, the sets $\mathcal{F}_{s,t}$ form a true two-parameter family, in that $\mathcal{F}_{s_1,t_1}$ and $\mathcal{F}_{s_2,t_2}$ are disjoint if $(s_1, t_1) \neq (s_2, t_2)$.

Remark 2.24. We remark that, if it is assumed that that (1.3) is globally well-posed in Y when $1 \leq p < 4/3$ (where p is rational with odd denominator), then the above stability result extends to these values of p as well, with the same proof.

Remark 2.25. From the definition of the variational problem for $W(s,t)$ it is clear that the sets $\mathcal{F}_{s,t}$ are invariant under the transformation

$$(\Phi(x), \psi(x)) \mapsto (e^{i\theta}\Phi(x - \xi), \psi(x - \xi)),$$

for every pair of real numbers θ and ξ, and so are at least two-dimensional in size. On the other hand, for a given solitary-wave profile (g, h) in $\mathcal{F}_{s,t}$, the orbit $\mathcal{O} = \{(u(x,t), v(x,t)) : t \in \mathbb{R}\}$ of the corresponding solitary wave is seen from

(1.12) to be given by

$$\mathcal{O} = \left\{ (e^{ict}\Phi(x-ct), \psi(x-ct)) : t \in \mathbb{R} \right\},$$

and hence is a proper (one-dimensional) subset of $\mathcal{F}_{s,t}$. Therefore Theorem 2.23 is somewhat weaker than an orbital stability result for the solitary waves in $\mathcal{F}_{s,t}$.

We now prove Theorem 2.23. To prove statement (i), we start from a given subsequence and use Lemma 2.22 to conclude that some subsequence of $(f_n, g_n) = (e^{ibx}h_n, g_n)$ is a minimizing sequence for $I(s,a)$. We claim that upon passing to a further subsequence, there exist real numbers y_n such that $(f_n(x+y_n), g_n(x+y_n))$ converges in Y to some (ϕ, ψ) in $\mathcal{S}_{s,a}$. If $a > 0$, this follows immediately from part (i) of Theorem 2.15. If, on the other hand, $a = 0$, then as in the proof of Lemma 2.22 we obtain (2.92). But from (2.92) we see that

$$\lim_{n\to\infty} E(f_n, g_n) = \lim_{n\to\infty} E(f_n, 0),$$

and since $E(f_n, g_n)$ converges to $I(s,0)$, this means that $(f_n, 0)$ is a minimizing sequence for $I(s,0)$. Since $a = 0$, then Lemma 2.22 implies that β_1 must be positive, so the claim follows from Lemma 2.7. Thus the claim has been proved in all cases.

Now, by passing to yet another subsequence, we may assume that e^{iby_n} converges to $e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Then $(h_n(.+y_n), g_n(.+y_n))$ converges to (Φ, ψ) in Y, where $\Phi(x) = e^{-i(bx+\theta)}\phi(x)$. As in (2.90), we have

$$\begin{aligned} I(s,a) = E(\phi,\psi) &= E(\Phi,\psi) + b^2 H(\Phi) - 2b \ \operatorname{Im} \int_{-\infty}^{\infty} \Phi\overline{\Phi}_x \, dx \\ &= E(\Phi,\psi) + b^2 s - 2b\left(G(\Phi,\psi) - \|\psi\|^2\right) \\ &= E(\Phi,\psi) + b^2 s - 2b(t-s) = E(\Phi,\psi) - b^2 s. \end{aligned} \tag{2.95}$$

It then follows from (2.89) that $E(\Phi, \psi) = W(s,t)$, and hence that $(\Phi, \psi) \in \mathcal{F}_{s,t}$.

Part (ii) of the Theorem follows from the Lagrange multiplier principle, just as did part (ii) of Theorem 2.15.

Next we prove part (iii) of Theorem 2.23. Suppose $(\Phi, \psi) \in \mathcal{F}_{s,t}$. Applying Lemma 2.22 to the minimizing sequence for $W(s,t)$ defined by setting $(h_n, g_n) = (\Phi, \psi)$ for all $n \in \mathbb{N}$, we obtain that $(e^{ibx}\Phi, \psi)$ is a minimizing sequence for $I(s,a)$, where $a = \|g\|^2$ and $b = (t-a)/s$. Therefore $(e^{ibx}\Phi, \psi) \in \mathcal{S}_{s,a}$. Hence by part (iii) of Theorem 2.15, there exist a number $\theta \in \mathbb{R}$ and a real-valued function $\tilde{\phi}$ such that $e^{ibx}\Phi(x) = e^{i\theta}\tilde{\phi}(x)$. So

$$\Phi(x) = e^{i(-bx+\theta)}\tilde{\phi}(x),$$

which is (2.94). In case $\tau_1 = 0$, then $\beta_1 = 0$ and it follows from Lemma 2.22 that $a > 0$. Since $(\tilde{\phi}, \psi) \in \mathcal{S}_{s,a}$, it follows from part (iii) of Theorem 2.15 that $\psi(x) > 0$ on $\mathbb{R}$, and that either $\tilde{\phi}(x) > 0$ for all $x \in \mathbb{R}$ or $\tilde{\phi}(x) < 0$ for all $x \in \mathbb{R}$. In the latter case, we can add π to the value of θ and replace $\tilde{\phi}$ by $e^{i\theta}\tilde{\phi}$ to get that $\tilde{\phi}$ is positive on $\mathbb{R}$.

To prove part (v) of Theorem (2.23), suppose that $\mathcal{F}_{s,t}$ is not stable. Then there exist a number $\epsilon > 0$ and sequences (h_n, g_n) of initial data in Y and times $t_n \geq 0$ such that, for all $n \in \mathbb{N}$,

$$\inf\{\|(h_n, g_n) - (h,g)\|_Y : (h,g) \in \mathcal{F}_{s,t}\} < \frac{1}{n}; \tag{2.96}$$

while the solutions $(u_n(x,t), v_n(x,t))$ of (1.3) with initial data

$$(u_n(x,0), v_n(x,0)) = (h_n(x), g_n(x))$$

satisfy

$$\inf\{\|(u_n(\cdot,t_n),v_n(\cdot,t_n)-(h,g)\|_Y : (h,g)\in\mathcal{F}_{s,t}\}\geq\epsilon \tag{2.97}$$

for all $n\in\mathbb{N}$.

From (2.96) and Lemma 2.11 we have that

$$\begin{aligned}
\lim_{n\to\infty} E(h_n,g_n) &= W(s,t),\\
\lim_{n\to\infty} H(h_n) &= s,\\
\lim_{n\to\infty} G(h_n,g_n) &= t.
\end{aligned} \tag{2.98}$$

Let us denote $u_n(\cdot,t_n)$ by U_n and $v_n(\cdot,t_n)$ by V_n. Since $E(u,v)$, $G(u,v)$, and $H(u)$ are independent of t, then (2.98) implies

$$\begin{aligned}
\lim_{n\to\infty} E(U_n,V_n) &= W(s,t),\\
\lim_{n\to\infty} H(U_n) &= s,\\
\lim_{n\to\infty} G(U_n,V_n) &= t,
\end{aligned}$$

which means that $\{(U_n,V_n)\}$, like $\{(h_n,g_n)\}$, is a minimizing sequence for $W(s,t)$.

Now part (i) of Theorem 2.23 tells us that there exists a subsequence $\{(U_{n_k},V_{n_k})\}$, a sequence of real numbers $\{y_k\}$, and a function pair $(\Phi,\psi)\in\mathcal{F}_{s,t}$ such that

$$\lim_{k\to\infty}\|(U_{n_k}(\cdot+y_k),V_{n_k}(\cdot+y_k))-(\Phi,\psi)\|_Y=0. \tag{2.99}$$

So, for some sufficiently large k,

$$\|(U_{n_k}(\cdot+y_k),V_{n_k}(\cdot+y_k))-(\Psi,\psi)\|_Y<\epsilon,$$

and hence

$$\|(U_{n_k},V_{n_k})-(\Phi(\cdot-y_k),\psi(\cdot-y_k))\|_Y<\epsilon. \tag{2.100}$$

But $(\Phi(\cdot - y_k), \psi(\cdot - y_k))$ is also in $\mathcal{F}_{s,t}$, and hence (2.100) gives

$$\inf\{\|(U_{n_k}, V_{n_k}) - (h,g)\|_Y : (h,g) \in \mathcal{F}_{s,t}\} < \epsilon.$$

Since this contradicts (2.97), we conclude that $\mathcal{F}_{s,t}$ must in fact be stable.

It remains only to prove that the sets $\mathcal{F}_{s,t}$ form a true two-parameter family. Suppose $(\Phi_1, \psi_1) \in \mathcal{F}_{s_1,t_1}$ and $(\Phi_2, \psi_2) \in \mathcal{F}_{s_2,t_2}$, where $(s_1, t_1) \neq (s_2, t_2)$. We want to show $(\Phi_1, \psi_1) \neq (\Phi_2, \psi_2)$. If $s_1 \neq s_2$, the conclusion is obvious, since then $\|\Phi_1\|^2 \neq \|\Phi_2\|^2$. So we can assume $s_1 = s_2$ and $t_1 \neq t_2$. From part (iii), if we let $\eta_i = (\|\psi_i\|^2 - t_1)/s_i$ for $i = 1, 2$; then there exist numbers θ_1 and θ_2 and real-valued functions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ such that

$$\Phi_1(x) = e^{i(\eta_1 x + \theta_1)}\tilde{\phi}_1(x) \quad \text{and} \quad \Phi_2(x) = e^{i(\eta_2 x + \theta_2)}\tilde{\phi}_2(x) \tag{2.101}$$

on $\mathbb{R}$. We may assume that $\Phi_1 = \Phi_2$, or else we are done. Then

$$e^{i((\eta_1 - \eta_2)x + (\theta_1 - \theta_2))} = \tilde{\phi}_2(x)/\tilde{\phi}_1(x)$$

is real-valued on $\mathbb{R}$, and hence η_1 must equal η_2. Since $s_1 = s_2$, this implies that $\|\psi_1\|^2 - t_1 = \|\psi_2\|^2 - t_2$. But $t_1 \neq t_2$, so therefore $\|\psi_1\|^2 \neq \|\psi_2\|^2$, and hence $\psi_1 \neq \psi_2$, as desired.

The proof of Theorem 2.23 is now complete.

Chapter 3

Stability of Solitary Waves–A Different Method

The techniques presented in Chapter 2 for proving stability of solitary waves works whenever the functionals involved in the variational analysis are conserved quantities for the evolution equation in question. In this chapter, we will show how the concentration compactness method can still be used to prove the stability of solitary waves if the functionals involved in the variational problem are not conserved quantities. By considering a different variational problem and using convexity techniques, we establish the stability result of solitary waves of (1.3) when $p = 1, q = 1$, and $\tau_2 = 1$.

3.1 Introduction

We consider the following nonlinear Schrödinger-KdV system

$$\begin{cases} iu_t + u_{xx} + \beta\,|u|\,u = -\alpha uv \\ v_t + v_{xxx} + vv_x = -\frac{\alpha}{2}\left(|u|^2\right)_x \end{cases}, \tag{3.1}$$

where $u = u(x,t) \in \mathbb{C}$ denotes the short wave term, $v = v(x,t) \in \mathbb{R}$ denotes the long wave term, and α, β are positive real constants. To obtain solitary-wave solutions of the system (3.1), we set

$$(u(x,t), v(x,t)) = (e^{i\omega t}e^{ic(x-ct)/2}\phi(x-ct),\ \psi(x-ct)), \tag{3.2}$$

and we may transform the system (3.1) to the following system of ODEs

$$\left\{ \begin{array}{l} -\phi'' + \sigma\phi = \beta\left|\phi\right|\phi + \alpha\psi\phi \\ -\psi'' + c\psi = \frac{1}{2}(\psi^2 + \alpha\phi^2) \end{array} \right. , \tag{3.3}$$

The conserved functionals H, G, and E for the system (3.1) are given by

$$H(u) = \int_{-\infty}^{\infty} \left|u\right|^2 dx, \tag{3.4}$$

$$G(u,v) = \int_{-\infty}^{\infty} v^2 \; dx + \operatorname{Im}\int_{-\infty}^{\infty} u\overline{u}_x \; dx, \tag{3.5}$$

$$E(u,v) = \int_{-\infty}^{\infty} \left(\left|u_x\right|^2 + v_x^2 - \frac{1}{3}v^3 - \frac{2\beta}{3}\left|u\right|^3 - \alpha v\left|u\right|^2 \right) dx. \tag{3.6}$$

One question we address below is whether nontrivial solutions of (3.1) exist. Our existence result is obtained by studying a different variational problem and using the concentration compactness method. We use the following three-step approach to prove the existence of travelling-wave solutions:

Step 1 : We consider first the problem of finding, for $\lambda > 0$,

$$I_\lambda = \inf\left\{ Z_{c,\omega}(f,g) : (f,g) \in X \text{ and } N(f,g) = \lambda \right\},$$

where $Z_{c,\omega}(f,g)$ and $N(f,g)$ are given by

$$Z_{c,\omega}(f,g) = \int_{-\infty}^{\infty} [(f'(x))^2 + \sigma f^2(x) + (g'(x))^2 + cg^2(x)] \; dx \tag{3.7}$$

with $c > 0, \sigma > 0$, and

$$N(f,g) = \int_{-\infty}^{\infty} \left[\alpha g(x) f^2(x) \; dx + \frac{2\beta}{3} f^3(x) + \frac{1}{3} g^3(x) \right] \; dx. \tag{3.8}$$

Using the concentration compactness method, we show that the set of minimizers

$$P_\lambda = \{(f,g) \in X : Z_{c,\omega}(f,g) = I_\lambda,\ N(f,g) = \lambda\}$$

is non-empty. Moreover, any minimizing sequence $\{(f_n, g_n)\}$ is compact in X up to translation.

Step 2 : We consider next the minimization problem over $Y := H^1_{\mathbb{C}} \times H^1$ and establish the relation between this complex case and the real case in Step 1. More precisely, we consider the following minimization problem

$$I^{\mathbb{C}}_\lambda = \inf\{Z^{\mathbb{C}}_{c,\omega}(h,g) : (h,g) \in Y \text{ and } \mathcal{N}(h,g) = \lambda\},$$

where $Z^{\mathbb{C}}_{c,\omega}(h,g)$ and $\mathcal{N}(h,g)$ are defined by

$$Z^{\mathbb{C}}_{c,\omega}(h,g) = \int_{-\infty}^{\infty} \left[|h'(x)|^2 + \sigma |h(x)|^2 + (g'(x))^2 + cg^2(x)\right]\ dx \tag{3.9}$$

with $c > 0, \sigma > 0$, and

$$\mathcal{N}(h,g) = \int_{-\infty}^{\infty} \left[\alpha g(x) |h(x)|^2 + \frac{2\beta}{3} |h(x)|^3 + \frac{1}{3} g^3(x)\right]\ dx. \tag{3.10}$$

Step 3 : Our theory of stability has another variational characterization of solitary-wave solutions for (3.1). For $c > 0$ and $\omega > c^2/4$, we consider the full minimization problem over Y,

$$J_\lambda = \inf\{Q_{c,\omega}(h,g) : (h,g) \in Y \text{ and } \mathcal{N}(h,g) = \lambda\},$$

where $\lambda > 0$, and $Q_{c,\omega}(h,g)$ is defined by

$$Q_{c,\omega}(h,g) = \int_{-\infty}^{\infty} \left[|h'|^2 + (g')^2 + \omega |h|^2 + cg^2 + c\,\mathrm{Im}(h\overline{h'}) \right] \, dx. \tag{3.11}$$

Then the set $\mathcal{P}_\lambda$ of minimizers of J_λ is non-empty, namely

$$\mathcal{P}_\lambda = \{(h,g) \in Y_{\mathbb{C}} : Q_{c,\omega}(h,g) = J_\lambda \text{ and } \mathcal{N}(h,g) = \lambda\} \neq \emptyset.$$

Moreover, any mimimizing sequence $\{(h_n, g_n)\}$ is compact in Y up to translation and rotation, that is, there are subsequences $\{(h_{n_k}, g_{n_k})\}$, $\{y_{n_k}\} \subset \mathbb{R}$ and $(h,g) \in \mathcal{P}_\lambda$ such that

$$\{(e^{icy_k/2} h_{n_k}(. - y_{n_k}), g_{n_k}(. - y_{n_k}))\}$$

converges strongly in Y to (h,g). Furthermore, $(h,g) = (e^{i\theta} e^{icx/2} f, g)$ where $\theta \in \mathbb{R}$ and $(f,g) \in P_\lambda$.

The three-step approach gives the existence of travelling-wave solutions to (3.1). For the stability theory, we make use of the functionals H, G, and E to obtain a relationship that makes it possible to utilize the variational properties of the traveling waves in the stability analysis. We show that the set of solitary waves is stable provided the associated action is strictly convex.

3.2 Existence of solitary waves

In this section we prove the existence of solitary-wave solutions for the equation (3.1) by using the concentration compactness method.

First, we consider the minimization problem over the real numbers, that is, the problem of finding, for any $\lambda > 0$,

$$I_\lambda = \inf \{Z_{c,\omega}(f,g) : (f,g) \in X \text{ and } N(f,g) = \lambda\}, \tag{3.12}$$

where $Z_{c,\omega}(f,g)$ and $N(f,g)$ are given by (3.7) and (3.8) respectively. The set of minimizers for I_λ is

$$P_\lambda = \{(f,g) \in X : Z_{c,\omega}(f,g) = I_\lambda,\ N(f,g) = \lambda\}$$

and the minimizing sequence for I_λ is any sequence $\{(f_n, g_n)\}$ of functions in X satisfying

$$\lim_{n\to\infty} Z_{c,\omega}(f_n, g_n) = I_\lambda \text{ and } N(f_n, g_n) = \lambda,\ \forall n.$$

Clearly, the functional $Z_{c,\omega}(f,g) \geq 0$ and hence I_λ is non-negative. It will be shown later that indeed $I_\lambda > 0$.

Remark 3.1. Because of the homogeneity of the functionals involved,

$$\inf\{Z_{c,\omega}(f,g) : N(f,g) = 1\} = \inf\{\tfrac{1}{\lambda^{2/3}} Z_{c,\omega}(f,g) : N(f,g) = \lambda\},$$

it follows that for any $\lambda > 0$, $I_\lambda = \lambda^{2/3} I_1$. Because of this homogeneity, we consider instead the problem

$$I_1 = \inf\{Z_{c,\omega}(f,g) : (f,g) \in X,\ N(f,g) = 1\}.$$

Let $\{(f_n, g_n)\}$ be a minimizing sequence and consider the concentration function $\rho_n(x) := (f_n')^2 + f_n^2 + (g_n')^2 + g_n^2$. As $\|(f_n, g_n)\|_X \leq C$ for all n, the sequence $\{a_n\}$ of real numbers given by

$$a_n := \int_{-\infty}^{\infty} \rho_n(x)\, dx$$

is bounded. Therefore, by passing to a subsequence, we may assume that a_n converges to a limit $a \in \mathbb{R}$. So by restricting consideration to the corresponding subsequence of ρ_n, which we again denote as ρ_n, we may assume that

$$a = \lim_{n\to\infty} \int_{-\infty}^{\infty} \rho_n(x)dx.$$

Define a sequence of nondecreasing functions $M_n : [0,\infty) \to [0,a]$ as follows

$$M_n(r) = \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} \rho_n(x)dx.$$

As $M_n(r)$ is a uniformly bounded sequence of nondecreasing function in r, it is straight-forward to show that it has a subsequence, which we still denote as M_n, that converges pointwise to a nondecreasing limit function $M(r) : [0,\infty) \to [0,a]$. Let

$$a_0 = \lim_{r\to\infty} M(r) :\equiv \lim_{r\to\infty} \lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} \rho_n(x)dx.$$

Then $0 \leq a_0 \leq a$. Our goal is to show that the possibilities $a_0 = 0$ and $a_0 \in (0,a)$, which correspond to the vanishing and dichotomy alternatives in Lions' Concentration Compactness Lemma, do not occur.

The following technical lemma is needed to rule out the case of vanishing.

Lemma 3.2. *There exists a constant $C > 0$ such that for every interval $I \subset \mathbb{R}$ of length 1 and every $g \in H^1(I)$, one has*

$$\left(\sup_{x\in I} |g(x)|\right)^2 \leq C \int_I \left[(g'(y))^2 + (g(y))^2\right] dy. \tag{3.13}$$

Proof. Let $I' = [0,1]$. By a standard Sobolev embedding theorem (sometimes called Morrey's inequality, see Theorem 5 of Section 5.6 of Evans [23]), there exists a constant C, independent of f, such that

$$\left(\sup_{x\in I'} |f(x)|\right)^2 \leq C \int_0^1 \left[(f'(y))^2 + (f(y))^2\right] dy \tag{3.14}$$

for all $f \in H^1(I')$ We claim that (3.13) holds on any interval of length 1 with the same constant C. Indeed, let $I = [a, a+1]$ be a given interval in $\mathbb{R}$ of length 1, and let $g \in H^1(I)$ be given. Then $f(x) := g(x-a)$ is in $H^1(I')$, so (3.14) applies, and hence

$$\left(\sup_{x\in I'} |g(x-a)|\right)^2 \leq C \int_0^1 \left[(g'(y-a))^2 + (g(y-a))^2\right] dy.$$

Then (3.13) follows immediately by a change of variables. $\square$

Lemma 3.3. *There exists a $\gamma > 0$ such that*

$$\lim_{n\to\infty} M_n\left(\frac{1}{2}\right) = \lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx \geq \gamma.$$

Therefore $a_0 \geq \gamma > 0$.

Proof. Suppose that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx = 0.$$

Let $j \in \mathbb{Z}$ be given, and let $I_j = [j-1/2, j+1/2]$. On the interval I_j, by Lemma 3.2, there exists a C (independent of j) such that

$$\begin{aligned}\left(\sup_{x\in I_j} |g_n(x)|\right)^2 &\leq C \int_{I_j} \left[(g_n'(y))^2 + (g_n(y))^2\right] dy \\ &\leq C \sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx,\end{aligned}$$

and also

$$\left(\sup_{x\in I_j} |f_n(x)|\right)^2 \leq C \sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx.$$

From the expression for $N(f,g)$, it is deduced that

$$
\begin{aligned}
|N(f_n,g_n)| &\leq \sum_{j=-\infty}^{\infty}\left(\sup_{x\in I_j}|g_n(x)|\right)\int_{I_j}\left[\alpha f_n^2(x)+\frac{1}{3}g_n^2(x)\right]dx \\
&\quad + \sum_{j=-\infty}^{\infty}\left(\sup_{x\in I_j}|f_n(x)|\right)\int_{I_j}\frac{2\beta}{3}f_n^2(x)dx \\
&\leq C\,\|(f_n,g_n)\|_X^2\left(\sup_{y\in\mathbb{R}}\int_{y-1/2}^{y+1/2}\rho_n(x)dx\right)^{1/2}.
\end{aligned}
$$

Hence $|N(f_n,g_n)|\to 0$ as $n\to\infty$, a contradiction. It follows that

$$
a_0=\lim_{r\to\infty}M(r)\geq M\left(\frac{1}{2}\right)=\lim_{n\to\infty}M_n\left(\frac{1}{2}\right)\geq\gamma>0,
$$

proving the lemma. □

From the preceding lemma it follows that $a_0\neq 0$ so that the sequence $\{\rho_n\}$ does not 'vanish' in the sense of Lions. Next we rule out the possibility of 'dichotomy'. To do this we need some preliminary results on the behavior of minimizing sequences in the case $0<a_0<a$.

Given any $\varepsilon>0$, for all sufficiently large values of r, we have

$$
a_0-\varepsilon<M(r)\leq M(2r)\leq a_0.
$$

Assume for the moment that such a value of r has been chosen. Then we can choose N large enough that

$$
a_0-\varepsilon<M_n(r)\leq M_n(2r)\leq a_0+\varepsilon
$$

for all $n \geq N$. Hence for each $n \geq N$ one can find y_n such that

$$\int_{y_n-r}^{y_n+r} \rho_n(x)dx > a_0 - \varepsilon, \text{ and} \int_{y_n-2r}^{y_n+2r} \rho_n(x)dx < a_0 + \varepsilon.$$

Choose $\phi \in C_0^\infty[-2,2]$ such that $\phi \equiv 1$ on $[-1,1]$, and let $\psi \in C^\infty(\mathbb{R})$ be such that $\phi^2 + \psi^2 \equiv 1$ on $\mathbb{R}$. For each $r \in \mathbb{R}$, let $\phi_r(x) = \phi(x/r)$ and $\psi_r(x) = \psi(x/r)$. Define

$$\begin{aligned} u_n(x) &= \phi_r(x-y_n)f_n(x), \quad \widetilde{u}_n(x) = \psi_r(x-y_n)f_n(x), \\ v_n(x) &= \phi_r(x-y_n)g_n(x), \quad \widetilde{v}_n(x) = \psi_r(x-y_n)g_n(x) \end{aligned}$$

and we consider

$$\rho_n^{(1)} = (u_n')^2 + u_n^2 + (v_n')^2 + v_n^2 \text{ and } \rho_n^{(2)} = (\widetilde{u}_n')^2 + \widetilde{u}_n^2 + (\widetilde{v}_n')^2 + \widetilde{v}_n^2.$$

Notice that $u_n, \widetilde{u}_n, v_n$ and $\widetilde{v}_n$ depend on r (which will be chosen later).

The following lemma is used to describe the behavior of $\{(f_n, g_n)\}$ in the case $0 < a_0 < a$ (the case of dichotomy).

Lemma 3.4. *For every $\varepsilon > 0$, there exist R and N large enough such that for all $n \geq N$ and $r \geq R$, one has*

(i) $Z_{c,\omega}(f_n, g_n) = Z_{c,\omega}(u_n, v_n) + Z_{c,\omega}(\widetilde{u}_n, \widetilde{v}_n) + O(\varepsilon)$

(ii) $N(f_n, g_n) = N(u_n, v_n) + N(\widetilde{u}_n, \widetilde{v}_n) + O(\varepsilon)$.

Proof. From the definition of $u_n, \widetilde{u}_n, v_n$ and $\widetilde{v}_n$, it follows that

$$\begin{aligned}
Z_{c,\omega}(u_n, v_n) + Z_{c,\omega}(\widetilde{u}_n, \widetilde{v}_n) &= \int_{-\infty}^{\infty} \left[(f_n')^2 + \sigma f_n^2 + (g_n')^2 + c g_n^2\right] \, dx \\
&\quad + \int_{-\infty}^{\infty} \left[\left((\phi_r')^2 + (\psi_r')^2\right)\left(f_n^2 + g_n^2\right)\right] \, dx \\
&\quad + \int_{-\infty}^{\infty} \left[(2\phi_r \phi_r' + 2\psi_r \psi_r')(f_n f_n' + g_n g_n')\right] \, dx,
\end{aligned}$$

where for brevity we have written simply ϕ_r and ψ_r for the functions $\phi_r(x - y_n)$ and $\psi_r(x - y_n)$. Now $|(\phi_r)'|_\infty = |\phi'|_\infty / r$ and $|(\psi_r)'|_\infty = |\psi'|_\infty / r$. Thus, making the use of Holder's Inequality, one can rewrite the preceding equation in the form

$$Z_{c,\omega}(u_n, v_n) + Z_{c,\omega}(\widetilde{u}_n, \widetilde{v}_n) = Z_{c,\omega}(f_n, g_n) + O\left(\frac{1}{r}\right),$$

where $O(1/r)$ denotes the term bounded in absolute value by A_1/r with A_1 independent of r and n. For $N(f_n, g_n)$, let us denote

$$B(f_n, g_n) := \alpha g_n f_n^2 + \frac{2\beta}{3} f_n^3 + \frac{1}{3} g_n^3.$$

Then we obtain

$$\begin{aligned}
N(u_n, v_n) + N(\widetilde{u}_n, \widetilde{v}_n) &= \int_{-\infty}^{\infty} B(f_n, g_n) \, dx \\
&\quad + \int_{-\infty}^{\infty} (\phi_r^3 - \phi_r^2 + \psi_r^3 - \psi_r^2) B(f_n, g_n) \, dx \\
&= N(f_n, g_n) + A_2 \varepsilon,
\end{aligned}$$

because

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty}(\phi_r^3-\phi_r^2+\psi_r^3-\psi_r^2)B(f_n,g_n)\right| &\le C\,|g_n|_\infty\left(\int_{r\le|x-y_n|\le 2r} f_n^2+g_n^2\right)\\
&\quad +C\,|f_n|_\infty\left(\int_{r\le|x-y_n|\le 2r} f_n^2\right)\\
&\le A_2\varepsilon,
\end{aligned}
$$

where again A_2 is independent of r and n. It is now time to choose r, and we make the choice so large that the $O(1/r)$ term is less than ε in absolute value. Consequently, for all $n \ge N$, we have

$$
Z_{c,\omega}(f_n,g_n)=Z_{c,\omega}(u_n,v_n)+Z_{c,\omega}(\tilde{u}_n,\tilde{v}_n)+O(\varepsilon)
$$

and

$$
N(f_n,g_n)=N(u_n,v_n)+N(\tilde{u}_n,\tilde{v}_n)+O(\varepsilon),
$$

proving the lemma. □

Lemma 3.5. $a_0 \notin (0,a)$, *the case of dichotomy cannot occur.*

Proof. The following argument is adapted from Levandosky [31]. Suppose dichotomy happens. Let $\{(f_n,g_n)\}$ be a minimizing sequence and consider the two sequences $\{(u_n,v_n)\}$ and $\{(\tilde{u}_n,\tilde{v}_n)\}$ as defined in Lemma 3.4. Then for large r, Lemma 3.4 assures that

$$
\begin{aligned}
Z_{c,\omega}(f_n,g_n) &= Z_{c,\omega}(u_n,v_n)+Z_{c,\omega}(\tilde{u}_n,\tilde{v}_n)+O(\varepsilon),\\
N(f_n,g_n) &= N(u_n,v_n)+N(\tilde{u}_n,\tilde{v}_n)+O(\varepsilon).
\end{aligned}
$$

As $\{(f_n,g_n)\}$ is bounded uniformly in X, it follows that $\{(u_n,v_n)\}$ and $\{(\tilde{u}_n,\tilde{v}_n)\}$ are also bounded independently of n and ε. Consequently $N(u_n,v_n)$ and $N(\tilde{u}_n,\tilde{v}_n)$

are bounded and we can pass to subsequences to define

$$\theta(\varepsilon) = \lim_{n\to\infty} N(u_n, v_n) \text{ and } \widetilde{\theta}(\varepsilon) = \lim_{n\to\infty} N(\tilde{u}_n, \tilde{v}_n).$$

As $\theta(\varepsilon)$ and $\widetilde{\theta}(\varepsilon)$ are bounded independently of ε, we can pick a sequence $\{\varepsilon_j\} \to 0$ such that both limits

$$\lim_{j\to\infty} \theta(\varepsilon_j) = \theta \;\text{ and }\; \lim_{j\to\infty} \widetilde{\theta}(\varepsilon_j) = \widetilde{\theta}$$

exist. Certainly, $\theta + \widetilde{\theta} = 1$, and there are 3 cases to consider now.

Case $1 : \theta \in (0,1)$. Then

$$\begin{aligned} Z_{c,\omega}(f_n, g_n) &= Z_{c,\omega}(u_n, v_n) + Z_{c,\omega}(\tilde{u}_n, \tilde{v}_n) + O(\varepsilon_j) \\ &\geq \left[N^{2/3}(u_n, v_n) + N^{2/3}(\tilde{u}_n, \tilde{v}_n)\right] I_1 + O(\varepsilon_j). \end{aligned}$$

We first let $n \to \infty$ to obtain

$$I_1 \geq \left[\theta^{2/3}(\varepsilon_j) + \widetilde{\theta}^{2/3}(\varepsilon_j)\right] I_1 + O(\varepsilon_j).$$

Then letting $j \to \infty$, we arrive at $I_1 \geq [\theta^{2/3} + \widetilde{\theta}^{2/3}]I_1 > I_1$, a contradiction.

Case $2 : \theta = 0$ (or equivalently, when $\theta = 1$), we have

$$\begin{aligned} Z_{c,\omega}(u_n, v_n) &\geq C\int_{-\infty}^{\infty} \left[(u_n')^2 + u_n^2 + (v_n')^2 + v_n^2\right]\, dx \\ &= C\int_{|x-y_n|\leq 2r} \left[(f_n')^2 + f_n^2 + (g_n')^2 + g_n^2\right]\, dx + O(\varepsilon_j) \\ &= Ca_0 + O(\varepsilon_j). \end{aligned}$$

Therefore

$$\begin{aligned} Z_{c,\omega}(f_n, g_n) &= Z_{c,\omega}(u_n, v_n) + Z_{c,\omega}(\tilde{u}_n, \tilde{v}_n) + O(\varepsilon_j) \\ &\geq Ca_0 + O(\varepsilon_j) + N^{2/3}(\tilde{u}_n, \tilde{v}_n) I_1. \end{aligned}$$

Letting n and $j \to \infty$ respectively, we obtain $I_1 \geq Ca_0 + I_1 > I_1$, which is a contradiction.

Case $3 : \theta > 1$ (or equivalently, when $\theta < 0$), we have

$$\begin{aligned} Z_{c,\omega}(f_n, g_n) &= Z_{c,\omega}(u_n, v_n) + Z_{c,\omega}(\tilde{u}_n, \tilde{v}_n) + O(\varepsilon_j) \\ &\geq Z_{c,\omega}(u_n, v_n) + O(\varepsilon_j) \geq N^{2/3}(u_n, v_n) I_1 + O(\varepsilon_j). \end{aligned}$$

As before, letting n and $j \to \infty$ respectively, we obtain that $I_1 \geq \theta^{2/3} I_1 > I_1$, a contradiction. Thus, each case gives a contradiction, which implies that $a_0 \notin (0, a)$. □

Theorem 3.6. *Let $\omega > c^2/4$ and λ be any positive number. Then any minimizing sequence $\{(f_n, g_n)\}$ for I_λ is relatively compact in X up to translation, i.e., there are subsequences $\{(f_{n_k}, g_{n_k})\}$ and $\{y_{n_k}\} \subset \mathbb{R}$ such that*

$$(f_{n_k}(. + y_{n_k}), g_{n_k}(. + y_{n_k}))$$

converges strongly in X to some (f, g) which is a minimum of I_λ. Therefore, the minimizing set P_λ is nonempty.

Proof. As we ruled out both vanishing and dichotomy, Lions' concentration compactness lemma ([33]) guarantees that sequence $\{\rho_n\}$ is tight, i.e., there exists a sequence of real numbers $\{y_n\}$ such that for any $\varepsilon > 0$, there exists

$r = r(\varepsilon)$ so that for all $n \in \mathbb{N}$,

$$\int_{y_n - r}^{y_n + r} \rho_n(x) dx = \int_{y_n - r}^{y_n - r} \left[(f_n')^2 + f_n^2 + (g_n')^2 + g_n^2 \right] \, dx > a - \varepsilon,$$

and therefore

$$\int_{-r}^{r} \rho_n(x + y_n) \, dx > a - \varepsilon. \tag{3.15}$$

Define w_n and z_n by $w_n(x) := f_n(x + y_n)$ and $z_n(x) := g_n(x + y_n)$. From (3.15), we have that

$$\int_{|x| \geq r(\varepsilon)} \left[(w_n)^2 + (z_n)^2 \right] dx < \varepsilon \tag{3.16}$$

for all $n \in \mathbb{N}$. Now since $\{(w_n, z_n)\}$ is bounded uniformly in X, there exists a subsequence, denoted again by $\{(w_n, z_n)\}$, which converges strongly in $L^2 \times L^2$ locally to some element (f, g) of X. We now show that $(w_n, z_n) \to (f, g)$ strongly in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Indeed, for any given $\varepsilon > 0$, we first choose r_0 so large that

$$\int_{|x| \geq r_0} \left[f^2(x) + g^2(x) \right] \, dx < \varepsilon. \tag{3.17}$$

Let $r_1 = \max\{r_0, r(\varepsilon)\}$. From (3.16) and (3.17), we have

$$\int_{|x| \geq r_1} \left[(f - w_n)^2 + (g - z_n)^2 \right] \, dx < 4\varepsilon.$$

On the other hand, from the strong convergence in $L^2 \times L^2$ locally of (w_n, z_n), there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N$,

$$\|(w_n, z_n) - (f, g)\|^2_{L^2(-r_1, r_1) \times L^2(-r_1, r_1)} < \varepsilon.$$

Consequently, $\|(w_n, z_n) - (f, g)\|^2_{L^2(\mathbb{R}) \times L^2(\mathbb{R})} < 5\varepsilon$, which proves that (w_n, z_n) converges strongly to (f, g) in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

Now, by the boundedness of w_n and z_n in H^1, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \left|w_n^2 z_n - f^2 g\right| \ dx &\leq \int_{-\infty}^{\infty} \left|w_n^2 z_n - w_n^2 g\right| \ dx + \int_{-\infty}^{\infty} \left|w_n^2 g - f^2 g\right| \ dx \\ &\leq C\left[\|w_n - f\|_{L^2(\mathbb{R})} + \|z_n - g\|_{L^2(\mathbb{R})}\right],\end{aligned}$$

and hence, $\int_{-\infty}^{\infty} w_n^2 z_n \ dx \to \int_{-\infty}^{\infty} f^2 g \ dx$.

Also, by the Sobolev embedding theorem,

$$|w_n - f|_3 \leq C \|w_n - f\|_1^{1/6} \|w_n - f\|^{5/6} \leq C \|w_n - f\|^{5/6},$$

so $\int_{-\infty}^{\infty} w_n^3 \ dx \to \int_{-\infty}^{\infty} f^3 \ dx$ as $n \to \infty$. Similarly, $\int_{-\infty}^{\infty} z_n^3 \ dx \to \int_{-\infty}^{\infty} g^3 \ dx$ as $n \to \infty$. Therefore, since $N(w_n, z_n) = \lambda$ for all n, it follows that $N(f,g) = \lambda$. Finally, by the weak compactness of the unit sphere in X and the weak lower semicontinuity of $Z_{c,\omega}$, we obtain

$$I_\lambda = \lim_{n\to\infty} Z_{c,\omega}(w_n, z_n) \geq Z_{c,\omega}(f,g).$$

Thus, (f,g) must be a minimizer for I_λ. □

Remark 3.7. Let $(f,g) \in P_1$. Then there exists some multiplier $K \in \mathbb{R}$ such that

$$\begin{cases} -f'' + \sigma f = K(\alpha f g + \beta f^2) \\ -g'' + cg = \frac{K}{2}(\alpha f^2 + g^2). \end{cases}$$

The Lagrange multiplier K is positive. Indeed, multiplying the first and second equations above by f and g, respectively, and then adding, we obtain that $K = 2I_1/3 > 0$.

We consider next the minimization problem with complex-valued functions ;

for $\lambda > 0$, let

$$I_\lambda^{\mathbb{C}} = \inf\{Z_{c,\omega}^{\mathbb{C}}(h,g) : (h,g) \in Y \ \text{ and } \mathcal{N}(h,g) = \lambda\}, \tag{3.18}$$

where $Z_{c,\omega}^{\mathbb{C}}(h,g)$ and $\mathcal{N}(h,g)$ are defined by (3.9) and (3.10) respectively. Clearly, $Z_{c,\omega}^{\mathbb{C}}(h,g) \geq 0$ and hence $I_\lambda^{\mathbb{C}}$ is non-negative. Notice that $Z_{c,\omega}^{\mathbb{C}}(h,g)$ is equivalent to the Y-norm of (h,g). Thus any minimizing sequence $\{(h_n,g_n)\}$ is uniformly bounded in Y. Then, by using exactly the same method as before with the concentration function $\rho_n = |h_n'|^2 + |h_n|^2 + (g_n')^2 + g_n^2$, the cases of vanishing and dichotomy of the minimizing sequence $\{(h_n,g_n)\}$ can be ruled out. Consequently, one has the following.

Theorem 3.8. *Let $\omega > c^2/4$ and λ be any positive number. Then, any minimizing sequence $\{(h_n,g_n)\}$ for $I_\lambda^{\mathbb{C}}$ is relatively compact in Y up to translation, i.e., there are subsequences $\{(h_{n_k},g_{n_k})\}$ and $\{y_{n_k}\} \subset \mathbb{R}$ such that*

$$(h_{n_k}(.+y_{n_k}), g_{n_k}(.+y_{n_k}))$$

converges strongly in Y to some (h,g) which is a minimum of $I_\lambda^{\mathbb{C}}$. Moreover, $(h,g) = (e^{i\theta}f, g)$ where $\theta \in \mathbb{R}$ and $(f,g) \in P_\lambda$.

Proof. The only thing that needs to be proved is the relation between minimizers. Let (h,g) be a minimizer of $I_\lambda^{\mathbb{C}}$ and let $h = h_1 + ih_2$. We claim that (h_0,g) is also a minimizer for $I_\lambda^{\mathbb{C}}$, where $h_0 = |h_1| + i\,|h_2|$. Indeed, using $\mathcal{N}(h_0,g) = \mathcal{N}(h,g) = \lambda$ and the inequality

$$\int_{-\infty}^{\infty} |h_i'(x)|^2 \ dx \geq \int_{-\infty}^{\infty} \left| |h_i|'(x) \right|^2 \ dx,$$

it follows that $I_\lambda^{\mathbb{C}} = Z_{c,\omega}^{\mathbb{C}}(h,g) \geq Z_{c,\omega}^{\mathbb{C}}(h_0,g) \geq I_\lambda^{\mathbb{C}}$, proving the claim. Therefore, there exists $K > 0$ (Lagrange's multiplier) such that

$$\begin{cases} -h_i'' + \sigma h_i = K(h_i g + |h_i|\, h_i) \\ -|h_i|'' + \sigma\,|h_i| = K(|h_i|\, g + |h|\,|h_i|) \end{cases}$$

for $i = 1, 2$. Since $|h_i| > 0$, it follows from the Sturm-Liouville theory that $-\sigma$ is the smallest eigenvalue of the operator $-\frac{d^2}{dx^2} - K(g + |h|)$ and therefore is simple. Hence there are $\mu_i \in \mathbb{R}\backslash\{0\}$ such that $h_i = \mu_i h_0^*$, where h_0^* is a positive function. Then $h = \mu_1 h_0^* + \mu_2 h_0^* = (\mu_1 + \mu_2) h_0^*$. Therefore, there exists a positive function f and $\theta \in \mathbb{R}$ such that $h = e^{i\theta} f$. Moreover, from the constraint $N(f, g) = \mathcal{N}(h, g) = \lambda$ and the fact that

$$I_\lambda^{\mathbb{C}} = Z_{c,\omega}^{\mathbb{C}}(h, g) = Z_{c,\omega}(f, g) \geq I_\lambda \geq I_\lambda^{\mathbb{C}},$$

one concludes that $(f, g) \in P_\lambda$. $\square$

The stability theory involves yet another variational characterization of solitary wave solutions for (3.1). For $\lambda > 0$, and fixed $c > 0$ and $\omega > c^2/4$, let

$$J_\lambda = \inf\{Q_{c,\omega}(h, g) : (h, g) \in Y \ \text{ and } \mathcal{N}(h, g) = \lambda\}, \tag{3.19}$$

where $Q_{c,\omega}(h, g)$ and $\mathcal{N}(h, g)$ are defined by (3.11) and (3.10) respectively. The set of minimizers of J_λ is

$$\mathcal{P}_\lambda = \{(h, g) \in Y : Q_{c,\omega}(h, g) = J_\lambda \ \text{ and } \mathcal{N}(h, g) = \lambda\},$$

and a minimizing sequence for J_λ is any sequence $\{(h_n, g_n)\}$ of functions in Y satisfying

$$\lim_{n\to\infty} Q_{c,\omega}(h_n, g_n) = J_\lambda \ \text{ and } \mathcal{N}(h_n, g_n) = \lambda, \ \forall n.$$

The next theorem gives the existence of a minimizer for J_λ and the relation between $\mathcal{P}_\lambda$ and P_λ.

Theorem 3.9. *Let $c > 0$, $\omega > c^2/4$ and λ be any positive number. Then the minimizing set $\mathcal{P}_\lambda$ is nonempty. Moreover, any minimizing sequence $\{(h_n, g_n)\}$ for J_λ is relatively compact in Y up to rotations and translation, that is, there are subsequences $\{(h_{n_k}, g_{n_k})\}$ and $\{y_{n_k}\} \subset \mathbb{R}$ such that*

$$(e^{icy_k/2} h_{n_k}(. + y_{n_k}), g_{n_k}(. + y_{n_k}))$$

converges strongly in Y to some (h, g) which is a minimum of J_λ Moreover, $(h, g) = (e^{i\theta} e^{icx/2} f, g)$ where $\theta \in \mathbb{R}$ and $(f, g) \in P_\lambda$.

Proof. Let $\{(h_n, g_n)\}$ be a minimizing sequence for J_λ. Then by definition,

$$\lim_{n\to\infty} Q_{c,\omega}(h_n, g_n) = J_\lambda \text{ and } \mathcal{N}(h_n, g_n) = \lambda.$$

Set $f_n = e^{-icx/2} h_n$, then one has $\mathcal{N}(f_n, g_n) = \mathcal{N}(h_n, g_n) = \lambda$ and

$$Q_{c,\omega}(h_n, g_n) = Q_{c,\omega}(e^{icx/2} f_n, g_n) = Z^{\mathbb{C}}_{c,\omega}(f_n, g_n) \geq I^{\mathbb{C}}_\lambda \tag{3.20}$$

Since $I^{\mathbb{C}}_\lambda \geq J_\lambda$, it follows from (3.20) that $\{(f_n, g_n)\}$ is a minimizing sequence for $I^{\mathbb{C}}_\lambda$. Therefore, by Theorem 3.8, there are subsequences $\{(f_{n_k}, g_{n_k})\}$ and $\{y_{n_k}\} \subset \mathbb{R}$ such that

$$(f_{n_k}(. + y_{n_k}), g_{n_k}(. + y_{n_k}))$$

converges strongly in Y to some (h_0, g) which is a minimum of $I^{\mathbb{C}}_\lambda$. Then $(h_0, g) = (e^{i\theta} f, g)$, where $\theta \in \mathbb{R}$ and $(f, g) \in P_\lambda$. Hence, from the definition of f_n, we have that

$$(e^{-icx/2} e^{icy_{n_k}/2} h_n, g_{n_k}(x + y_{n_k})) \to (e^{i\theta} e^{-icx/2} f, g)$$

in Y, and hence $(h, g) = (e^{i\theta} e^{icx/2} f, g) \in P_\lambda$. $\square$

Corollary 3.10. *Let $c > 0$, $\omega > c^2/4$ and λ be any positive number. If $\{(h_n, g_n)\}$ is any minimizing sequence for J_λ, then*

(i) $\lim\limits_{n\to\infty} \inf\limits_{\theta, y \in \mathbb{R}\ ;\ \overrightarrow{\psi} \in \mathcal{P}_\lambda} \| (e^{i\theta} h_n(. + y), g_n(. + y)) - \overrightarrow{\psi} \|_Y = 0$

(ii) $\lim\limits_{n\to\infty} \inf\limits_{\overrightarrow{\psi} \in \mathcal{P}_\lambda} \| (h_n, g_n) - \overrightarrow{\psi} \|_Y = 0.$

Proof. Suppose that (i) does not hold. Then there exists a subsequence $\{(h_{n_k}, g_{n_k})\}$ of $\{(h_n, g_n)\}$ and a number $\varepsilon > 0$ such that

$$\inf_{\theta, y \in \mathbb{R}\ ;\ \overrightarrow{\psi} \in \mathcal{P}_\lambda} \| (e^{i\theta} h_{n_k}(. + y), g_{n_k}(. + y)) - \overrightarrow{\psi} \|_Y \geq \varepsilon$$

for all $k \in \mathbb{N}$. But, since $\{(h_{n_k}, g_{n_k})\}$ itself is a minimizing sequence for $\mathcal{P}_\lambda$, Theorem 3.9 implies that there exists sequences $\{y_{n_k}\}$, $\{\theta_{n_k}\}$ and $\overrightarrow{\psi} \in \mathcal{P}_\lambda$ such that

$$\liminf_{k\to\infty}\ \inf \| (e^{i\theta_{n_k}} h_{n_k}(. + y_{n_k}), g_{n_k}(. + y_{n_k})) - \overrightarrow{\psi} \|_Y = 0,$$

which is a contradiction, and hence the statement (i) is proved.
Because of the invariance of the functionals $Q_{c,\omega}$ and $\mathcal{N}$ under rotations and translations, $\mathcal{P}_\lambda$ contains any rotations and translations of $\overrightarrow{\psi}$, if it contains $\overrightarrow{\psi}$. Consequently, statement (ii) follows from statement (i). $\square$

3.3 Stability of solitary-wave solutions

In this section we prove that the set of solitary waves is stable provided the associated action is strictly convex. We first establish some technical preliminaries that will be used in the stability analysis. Define the minimization problem

$$T_c(\omega) = \inf_{(h,g)\in Y} \frac{Q_{c,\omega}(h, g)}{[\mathcal{N}(h, g)]^{2/3}}. \tag{3.21}$$

Then, because of the homogeneity of the functionals involved, we have

$$T_c(\omega) = \inf_{(h,g)\in Y} \left\{ Q_{c,\omega}(h,g) : \mathcal{N}(h,g) = 1 \right\}. \tag{3.22}$$

Thus, if $(h,g) \in Y$ and satisfies $Q_{c,\omega}(h,g) = T_c(\omega)$ and $\mathcal{N}(h,g) = 1$, then from Theorem 3.9, $(h,g) = (e^{i\theta}e^{icx/2}f, g)$ where $\theta \in \mathbb{R}$ and $(f,g) \in P_1$. Hence, $T_c(\omega) = Q_{c,\omega}(h,g) = Z_{c,\omega}(f,g)$ and $N(f,g) = 1$, and so (3.22) can be written as

$$T_c(\omega) = \inf_{(h,g)\in X} \left\{ Z_{c,\omega}(f,g) : N(f,g) = 1 \right\} = I_1. \tag{3.23}$$

For fixed $c > 0$ and $\omega > c^2/4$, we define

$$\mathcal{B}_{c,\omega} = \left\{ (e^{i\theta}e^{icx/2}\phi, \psi) : (\phi,\psi) \in X,\ N(\phi,\psi) = \frac{2}{3} Z_{c,\omega}(\phi,\psi) = \frac{8}{27}[T_c(\omega)]^3 \right\}$$

Then, for $(e^{i\theta}e^{icx/2}\phi, \psi) \in \mathcal{B}_{c,\omega}$, we have that (ϕ,ψ) satisfies (3.3). Indeed, let $N(\phi,\psi) = \lambda$. Then, $N(\phi/\lambda^{1/3}, \psi/\lambda^{1/3}) = 1$ and further,

$$Q_{c,\omega}(e^{i\theta}e^{icx/2}\frac{1}{\lambda^{1/3}}\phi, \frac{1}{\lambda^{1/3}}\psi) = Z_{c,\omega}(\frac{1}{\lambda^{1/3}}\phi, \frac{1}{\lambda^{1/3}}\psi) = \frac{Z_{c,\omega}(\phi,\psi)}{[N(\phi,\psi)]^{2/3}} = T_c(\omega).$$

Therefore, $(e^{icx/2}\frac{1}{\lambda^{1/3}}\phi, \frac{1}{\lambda^{1/3}}\psi) \in \mathcal{P}_1$, and this implies $(\frac{1}{\lambda^{1/3}}\phi, \frac{1}{\lambda^{1/3}}\psi) \in P_1$. Then there is $K_0 \in \mathbb{R}$ such that

$$\begin{cases} -\phi'' + \sigma\phi = \frac{1}{\lambda^{1/3}} K_0(\alpha\phi\psi + \beta\phi^2) \\ -\psi'' + c\psi = \frac{1}{\lambda^{1/3}} \frac{K_0}{2}(\alpha\phi^2 + \psi^2). \end{cases}$$

Hence $Z_{c,\omega}(\phi,\psi) = \frac{3}{2}\frac{1}{\lambda^{1/3}} K_0 N(\phi,\psi)$, and so $K_0/\lambda^{1/3} = 1$, proving the claim.

Next, for $(\Phi_{c,\omega}(\xi), \Psi_{c,\omega}(\xi)) = (e^{ic\xi/2}\phi_{c,\omega}, \psi_{c,\omega}) \in \mathcal{B}_{c,\omega}$, we define the following functional

$$d(c,\omega) = E(\Phi_{c,\omega},\Psi_{c,\omega}) + \omega H(\Phi_{c,\omega}) + cG(\Phi_{c,\omega},\Psi_{c,\omega}), \tag{3.24}$$

where H, G and E are invariants of motion for (3.1), defined by (3.4), (3.5) and (3.6), respectively, and we consider the following function of one parameter ω, $d_c(\omega) \equiv d(c,\omega)$ with $c > 0$ fixed and $\omega \in (c^2/4, \infty)$. Then

$$\begin{aligned} d_c(\omega) &= Q_{c,\omega}(e^{ic\xi/2}\phi_{c,\omega}, \psi_{c,\omega}) - \mathcal{N}(e^{ic\xi/2}\phi_{c,\omega}, \psi_{c,\omega}) \\ &= Z_{c,\omega}(\phi_{c,\omega}, \psi_{c,\omega}) - N(\phi_{c,\omega}, \psi_{c,\omega}) \\ &= \frac{3}{2}N(\phi_{c,\omega}, \psi_{c,\omega}) - N(\phi_{c,\omega}, \psi_{c,\omega}) = \frac{1}{2}N(\phi_{c,\omega}, \psi_{c,\omega}) = \frac{4}{27}[T_c(\omega)]^3. \end{aligned}$$

Hence, the function $d_c(.)$ is well defined. Our goal is to show that the set of solitary waves $\mathcal{B}_{c,\omega}$ is stable with respect to (3.1) if $d_c(\omega)$ is a strictly convex function in ω.

Lemma 3.11. *$d_c(.)$ is well-defined on $(c^2/4, \infty)$, continuous, strictly increasing and is differentiable at all but countably many points of $(c^2/4, \infty)$.*

Proof. For nonzero fixed $(f,g) \in X$, $Z_{c,\omega}(f,g)/[N(f,g)]^{2/3}$ is just a line. Since $T_c(\omega)$ is the infimum of this family of lines, it follows that $T_c(\omega)$ is a concave function on $(c^2/4, \infty)$, and thus $T_c(\omega)$ is continuous and differentiable at all but countably many points and hence, we can conclude that same regularity properties hold for the function $d_c(.)$.

Now, let $c > 0$ fixed and $\omega_1 > \omega_2 > c^2/4$, let $(\Phi_1, \Psi_1) \in \mathcal{B}_{c,\omega_1}$ and $(\Phi_2, \Psi_2) \in \mathcal{B}_{c,\omega_2}$. Then

$$T_c(\omega_1) = \frac{Z_{c,\omega}(\phi_1,\psi_1)}{N^{2/3}(\phi_1,\psi_1)} \geq T_c(\omega_2) + \frac{(\omega_1-\omega_2)\int_{-\infty}^{\infty}\phi_1^2 dx}{N^{2/3}(\phi_1,\psi_1)} > T_c(\omega_2), \tag{3.25}$$

This shows that $T_c(\omega)$ is strictly increasing, so that $d_c(.)$ must be strictly increasing as well. $\square$

Remark 3.12. Let $\omega_1 > \omega_2$ and $(\Phi_1, \Psi_1), (\Phi_2, \Psi_2)$ as in Lemma 3.11. By reversing the roles of ω_1 and ω_2 in (3.25), we also have

$$T_c(\omega_2) = \frac{Z_{c,\omega}(\phi_2, \psi_2)}{N^{2/3}(\phi_2, \psi_2)} \geq T_c(\omega_1) + \frac{(\omega_2 - \omega_1) \int_{-\infty}^{\infty} \phi_2^2 dx}{N^{2/3}(\phi_2, \psi_2)}, \tag{3.26}$$

From (3.25) and (3.26), we find that

$$\frac{\int_{-\infty}^{\infty} \phi_1^2 dx}{N^{2/3}(\phi_1, \psi_1)} \leq \frac{T_c(\omega_1) - T_c(\omega_2)}{\omega_1 - \omega_2} \leq \frac{\int_{-\infty}^{\infty} \phi_2^2 dx}{N^{2/3}(\phi_2, \psi_2)}.$$

Because this holds for all solitary-wave solutions, we also have

$$\frac{9\alpha_c(\omega_1)}{4[T_c(\omega_1)]^2} \leq \frac{T_c(\omega_1) - T_c(\omega_2)}{\omega_1 - \omega_2} \leq \frac{9\beta_c(\omega_2)}{4[T_c(\omega_2)]^2}, \tag{3.27}$$

where $\alpha_c(\omega)$ and $\beta_c(\omega)$ are the infimum and supremum, respectively, of

$$\left\{ \int_{\mathbb{R}} |\Phi(x)|^2 : (\Phi, \Psi) \in \mathcal{B}_{c,\omega} \right\}.$$

At points of differentiability, we have $\alpha_c(\omega) = \beta_c(\omega)$ (see Lemma 3.2, [31]), and hence $d_c'(\omega) = H(\Phi) = H(\phi)$.

The following result is taken from Shatah [41].

Lemma 3.13. *Let h be any function which is strictly convex in an interval I around ω. Then given $\varepsilon > 0$, there exist $N(\varepsilon) > 0$ such that for $\omega_1 \in I$ and $|\omega_1 - \omega| \geq \varepsilon$, we have* *(i) For* $\omega_1 < \omega < \omega_0,\ |\omega_0 - \omega| < \varepsilon/2,\ \omega_0 \in I$. *Then*

$$\frac{h(\omega_1) - h(\omega_0)}{\omega_1 - \omega_0} \leq \frac{h(\omega) - h(\omega_0)}{\omega - \omega_0} - \frac{1}{N(\varepsilon)}.$$

(ii) For $\omega_0 < \omega < \omega_1,\ |\omega_0 - \omega| < \varepsilon/2,\ \omega_0 \in I$. *Then*

$$\frac{h(\omega_1) - h(\omega_0)}{\omega_1 - \omega_0} \geq \frac{h(\omega) - h(\omega_0)}{\omega - \omega_0} + \frac{1}{N(\varepsilon)}.$$

From Lemma 3.13 and from the inequalities (3.27), the following result holds.

Lemma 3.14. *Suppose that* $d_c(.)$ *is strictly convex in an interval* I *around* ω. *Then given* $\varepsilon > 0$, *there exists* $N(\varepsilon) > 0$ *such that for* $\omega_1 \in I$ *and* $|\omega_1 - \omega| \geq \varepsilon$, *we have*

$$\begin{aligned} d_c(\omega_1) &\geq d_c(\omega) + \beta_c(\omega)(\omega_1 - \omega) + \frac{1}{N(\varepsilon)}(\omega - \omega_1) \text{ for } \omega_1 < \omega; \\ d_c(\omega_1) &\geq d_c(\omega) + \alpha_c(\omega)(\omega_1 - \omega) + \frac{1}{N(\varepsilon)}(\omega_1 - \omega) \text{ for } \omega_1 > \omega. \end{aligned}$$

For $\varepsilon > 0$, we define the following $\varepsilon-$neighborhood of the set $\mathcal{B}_{c,\omega}$,

$$U_{c,\omega,\varepsilon} = \left\{ (u,v) \in Y : \inf_{(\Phi,\Psi)\in\mathcal{B}_{c,\omega}} \|(u,v) - (\Phi,\Psi)\|_Y < \varepsilon \right\}.$$

From the facts that $d_c(.)$ is a continuous strictly increasing, $\mathcal{B}_{c,\omega}$ is a bounded set in Y and the function $(\Phi,\Psi) \mapsto \mathcal{N}(\Phi,\Psi)$ is uniformly continuous on the bounded set, it is deduced that there is a small ε and a continuous map $\rho : U_{c,\omega,\varepsilon} \to (c^2/4, \infty)$, defined by

$$\rho(u,v) = d_c^{-1}\left(\frac{1}{2}\mathcal{N}(u,v)\right) = d_c^{-1}\left(\frac{1}{2}N(u,v)\right) \tag{3.28}$$

such that $\rho(\Phi_{c,\omega}, \Psi_{c,\omega}) = \omega$ for any $(\Phi_{c,\omega}, \Psi_{c,\omega}) \in \mathcal{B}_{c,\omega}$.

The following lemma is needed in proving our stability result.

Lemma 3.15. *Let $\omega > c^2/4$ and a fixed $c > 0$. Suppose that d_c is strictly convex in an interval I around ω. Then there exists $\varepsilon > 0$ such that for any $\overrightarrow{u} = (u, v) \in U_{c,\omega,\varepsilon}$ and any $\overrightarrow{\Phi} = (\Phi_{c,\omega}, \Psi_{c,\omega}) \in \mathcal{B}_{c,\omega}$, one has*

$E(\overrightarrow{u}) - E(\overrightarrow{\Phi}) + \rho(\overrightarrow{u})(H(\overrightarrow{u}) - H(\overrightarrow{\Phi})) + c(G(\overrightarrow{u}) - G(\overrightarrow{\Phi})) \geq \frac{1}{N(\varepsilon)} \left|\rho(\overrightarrow{u}) - \omega\right|$,

where $\rho(\overrightarrow{u})$ is defined in (3.28) and $N(\varepsilon)$ is given by Lemma 3.14.

Proof. Let ε be small enough such that $\rho(U_{c,\omega,\varepsilon}) \subset (\omega - \eta, \infty) \subset (c^2/4, \infty)$ for $\eta > 0$. Then, since

$$E(\overrightarrow{u}) + \rho(\overrightarrow{u})H(\overrightarrow{u}) + cG(\overrightarrow{u}) = Q_{c,\rho(\overrightarrow{u})}(\overrightarrow{u}) - \mathcal{N}(\overrightarrow{u}), \tag{3.29}$$

$2d_c(\rho(\overrightarrow{u})) = \mathcal{N}(\overrightarrow{u})$ and $\mathcal{N}(\Phi_{c,\rho(\overrightarrow{u})}, \Psi_{c,\rho(\overrightarrow{u})}) = 2d_c(\rho(\overrightarrow{u}))$, we get that $\mathcal{N}(\overrightarrow{u}) = \mathcal{N}(\Phi_{c,\rho(\overrightarrow{u})}, \Psi_{c,\rho(\overrightarrow{u})})$. Therefore

$$Q_{c,\rho(\overrightarrow{u})}(\overrightarrow{u}) \geq Q_{c,\rho(\overrightarrow{u})}(\Phi_{c,\rho(\overrightarrow{u})}, \Psi_{c,\rho(\overrightarrow{u})}) \tag{3.30}$$

and hence, from (3.29), (3.30) and using Lemma 3.14, we obtain

$$E(\overrightarrow{u}) + \rho(\overrightarrow{u})H(\overrightarrow{u}) + cG(\overrightarrow{u}) = Q_{c,\rho(\overrightarrow{u})}(\overrightarrow{u}) - \mathcal{N}(\overrightarrow{u})$$

$$\begin{aligned}
&\geq \; Q_{c,\rho(\overrightarrow{u})}(\Phi_{c,\rho(\overrightarrow{u})}, \Psi_{c,\rho(\overrightarrow{u})}) - \mathcal{N}(\Phi_{c,\rho(\overrightarrow{u})}, \Psi_{c,\rho(\overrightarrow{u})}) = d_c(\rho(\overrightarrow{u})) \\
&\geq \; d_c(\omega) + H(\overrightarrow{\Phi})(\rho(\overrightarrow{u}) - \omega) + \frac{1}{N(\varepsilon)} \left|\rho(\overrightarrow{u}) - \omega\right| \\
&= \; E(\overrightarrow{\Phi}) + cG(\overrightarrow{\Phi}) + \rho(\overrightarrow{u})H(\overrightarrow{\Phi}) + \frac{1}{N(\varepsilon)} \left|\rho(\overrightarrow{u}) - \omega\right|.
\end{aligned}$$

This proves the Lemma. ∎

The following is our stability theorem. It gives a sufficient condition for the set $\mathcal{B}_{c,\omega}$ of solitary waves to be stable with respect to (3.1).

Theorem 3.16. *Let $c > 0$ be fixed, $\omega > c^2/4$ and suppose that d_c is strictly convex in an interval I around ω. Then the set $\mathcal{B}_{c,\omega}$ of solitary waves is $Y-$stable, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$\inf_{(\Phi_{c,\omega},\Psi_{c,\omega})\in\mathcal{B}_{c,\omega}} \|(u_0, v_0) - (\Phi_{c,\omega}, \Psi_{c,\omega})\|_Y < \delta,$$

then the solution $(u(x,t), v(x,t))$ of (3.1) (or to (3.3)) with $(u(x,0), v(x,0)) = (u_0, v_0)$ satisfies

$$\inf_{(\Phi_{c,\omega},\Psi_{c,\omega})\in\mathcal{B}_{c,\omega}} \|(u(.,t), v(.,t)) - (\Phi_{c,\omega}, \Psi_{c,\omega})\|_Y < \varepsilon, \quad \forall t \geq 0.$$

Proof. Suppose the Theorem is false. Choose initial data $\overrightarrow{u}_k(0) \in U_{c,\omega,1/k}$ such that

$$\sup_{0\leq t<\infty} \inf_{\Phi\in\mathcal{B}_{c,\omega}} \left\|\overrightarrow{u}_k(t) - \overrightarrow{\Phi}\right\|_Y \geq \delta,$$

where $\overrightarrow{u}_k(t) = (u_k(t), v_k(t))$ is a solution of (3.1) with initial data $\overrightarrow{u}_k(0)$. Then, by the continuity in t, there exists t_k such that

$$\inf_{\Phi\in\mathcal{B}_{c,\omega}} \left\|\overrightarrow{u}_k(t_k) - \overrightarrow{\Phi}\right\|_Y = \delta. \tag{3.31}$$

Since E, H, G are invariants of (3.1) and since $\mathcal{B}_{c,\omega}$ is bounded, we can find $\overrightarrow{\Phi}_k \in \mathcal{B}_{c,\omega}$ such that

$$\begin{aligned}
\left|E(\overrightarrow{u}_k(t_k)) - E(\overrightarrow{\Phi}_k)\right| &= \left|E(\overrightarrow{u}_k(0)) - E(\overrightarrow{\Phi}_k)\right| \to 0,\\
\left|H(\overrightarrow{u}_k(t_k)) - H(\overrightarrow{\Phi}_k)\right| &= \left|H(\overrightarrow{u}_k(0)) - H(\overrightarrow{\Phi}_k)\right| \to 0,\\
\left|G(\overrightarrow{u}_k(t_k)) - G(\overrightarrow{\Phi}_k)\right| &= \left|G(\overrightarrow{u}_k(0)) - G(\overrightarrow{\Phi}_k)\right| \to 0.
\end{aligned}$$

as $k \to \infty$. If δ is chosen so small that Lemma 3.15 applies, then

$$\begin{aligned}
&E(\overrightarrow{u}_k(t_k)) - E(\overrightarrow{\Phi}_k) + \rho(\overrightarrow{u}_k(t_k)) \left[H(\overrightarrow{u}_k(t_k)) - H(\overrightarrow{\Phi}_k)\right] \\
&+c\left[G(\overrightarrow{u}_k(t_k)) - G(\overrightarrow{\Phi}_k)\right] \geq \frac{1}{N(\varepsilon)} \left|\rho(\overrightarrow{u}_k(t_k)) - \omega\right|
\end{aligned}$$

Since $\overrightarrow{u}_k(t_k)$ is uniformly bounded for k, so from the last inequality $\rho(\overrightarrow{u}_k(t_k)) \to \omega$ as $k \to \infty$. Hence, by (3.28) and the continuity of d_c, we have $\mathcal{N}(\overrightarrow{u}_k(t_k)) \to 2d_c(\omega)$ as $k \to \infty$. On the other hand, we have

$$\begin{aligned}
Q_{c,\omega}(\overrightarrow{u}_k(t_k)) &= d_c(\omega) + E(\overrightarrow{u}_k(t_k)) - E(\overrightarrow{\Phi}_k) + c[G(\overrightarrow{u}_k(t_k)) - G(\overrightarrow{\Phi}_k)] \\
&\quad +\omega[H(\overrightarrow{u}_k(t_k)) - H(\overrightarrow{\Phi}_k)] + \mathcal{N}(\overrightarrow{u}_k(t_k)) \ ;
\end{aligned}$$

consequently,

$$\lim_{k\to\infty} Q_{c,\omega}(\overrightarrow{u}_k(t_k)) = 3d_c(\omega) = \frac{4}{9}[T_c(\omega)]^3.$$

Let $\overrightarrow{w}_k(t_k) = [\mathcal{N}(\overrightarrow{u}_k(t_k))]^{-1/3}\overrightarrow{u}_k(t_k)$. Then $\mathcal{N}(\overrightarrow{w}_k(t_k)) = 1$ and

$$\begin{aligned}
\lim_{k\to\infty} Q_{c,\omega}(\overrightarrow{w}_k(t_k)) &= \lim_{k\to\infty} [\mathcal{N}(\overrightarrow{u}_k(t_k))]^{-2/3} Q_{c,\omega}(\overrightarrow{u}_k(t_k)) \\
&= \left(\frac{1}{2d_c(\omega)}\right)^{2/3} \frac{4}{9}[T_c(\omega)]^3 = T_c(\omega).
\end{aligned}$$

Therefore, $\overrightarrow{w}_k(t_k)$ is a minimizing sequence for J_1 and by Corollary 3.10, there exists $\overrightarrow{\psi}_k \in \mathcal{P}_1$ such that

$$\lim_{k\longrightarrow\infty} \left\|\overrightarrow{w}_k(t_k) - \overrightarrow{\psi}_k\right\|_Y = 0. \tag{3.32}$$

Now from Theorem 3.9, $\overrightarrow{\psi}_k = (e^{icx/2} f_k, g_k)$ for $(f_k, g_k) \in \mathcal{P}_1$, hence there exists

$K > 0$ such that

$$J(f_k, g_k) = \frac{3}{2}K.N(f_k, g_k) \Longrightarrow T_c(\omega) = \frac{3}{2}K \Longrightarrow K = \frac{2}{3}T_c(\omega).$$

Let $\lambda \widetilde{f}_k = f_k$ and $\lambda \widetilde{g}_k = g_k$. Then $J(\widetilde{f}_k, \widetilde{g}_k) = \frac{3}{2}\lambda K.N(\widetilde{f}_k, \widetilde{g}_k)$. Choosing $\lambda = 1/K$, we obtain $J(\widetilde{f}_k, \widetilde{g}_k) = \frac{3}{2}N(\widetilde{f}_k, \widetilde{g}_k)$ and

$$J(\widetilde{f}_k, \widetilde{g}_k) = \frac{1}{\lambda^2}J(f_k, g_k) = K^2 J(f_k, g_k) = \frac{4}{9}[T_c(\omega)]^3,$$

so that $N(\widetilde{f}_k, \widetilde{g}_k) = \frac{2}{3}J(\widetilde{f}_k, \widetilde{g}_k) = \frac{8}{27}[T_c(\omega)]^3$. Therefore, $(e^{icx/2}\widetilde{f}_k, \widetilde{g}_k) \in \mathcal{B}_{c,\omega}$. It follows that $\overrightarrow{\Psi}_k := (e^{icx/2}f_k, g_k) \in \mathcal{B}_{c,\omega}$, and so from (3.32), we have

$$0 = \lim_{k\to\infty} \left\| \overrightarrow{w}_k(t_k) - \lambda \overrightarrow{\Psi}_k \right\|_Y = \lim_{k\to\infty} \left\| \overrightarrow{w}_k(t_k) - \frac{3}{2}[T_c(\omega)]^{-1}\overrightarrow{\Psi}_k \right\|_Y \qquad (3.33)$$

Therefore, from (3.33) and since $\mathcal{B}_{c,\omega}$ being a bounded set in Y, we have

$$\begin{aligned}
\left\| \overrightarrow{u}_k(t_k) - \overrightarrow{\Psi}_k \right\|_Y &= |\mathcal{N}(\overrightarrow{u}_k(t_k))|^{1/3} \left\| [\mathcal{N}(\overrightarrow{u}_k(t_k))]^{-1/3}(\overrightarrow{u}_k(t_k) - \overrightarrow{\Psi}_k) \right\|_Y \\
&\leq |\mathcal{N}(\overrightarrow{u}_k(t_k))|^{1/3} \left[\left\| \overrightarrow{w}_k(t_k) - \frac{3}{2}[T_c(\omega)]^{-1}\overrightarrow{\Psi}_k \right\|_Y \right. \\
&\quad \left. + C \left| [\mathcal{N}(\overrightarrow{u}_k(t_k))]^{-1/3} + \frac{3}{2}[T_c(\omega)]^{-1} \right| \right]
\end{aligned}$$

and therefore we have that $\| \overrightarrow{u}_k(t_k) - \overrightarrow{\Psi}_k \|_Y \to 0$ as $k \to \infty$, which is a contradiction by (3.31). This completes the proof. $\square$

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